Wind Waves: Notes

Saint Mary's College
Chris Ray
Spring 2019

Contents

1 Introduction: Kinsman Ch1

§ 1.1 Introduction

▷ Problem 1.1
Define the following words.
(a) Fetch
(b) Dispersive medium.

▷ Problem 1.2
What is the most common height of waves on the ocean?

▷ Problem 1.3
On page page 15 are two equations one relating the wavelength L and period T and another relating the the wave speed c and the wavelength. Defining the wave number $k \equiv \frac{2\pi}{L}$ and frequency $\omega = \frac{2\pi}{T}$, and use these to replace the wavelength and period in the two equations. Simplify the two equations you get so that they looks pretty. Combine the two to show that $\frac{\omega}{k} = c$.

2 Fluids

This chapter deals with fluids. In our applications fluids will be in the form of liquids and gases. Liquids and gases behave differently than the solid objects we have dealt with so far. However, modern physical theory recognizes that all matter, whether solid, liquid or gas is ultimately composed of the same types of particles. It is only the arrangement of the particles that determines if they will behave together as a liquid, solid, or gas. A rough, qualitative description of the three different arrangements of matter is to describe the internal forces between the particles: A gas has very weak interactions between the particles, while a liquid has weak to moderate forces between particles, and finally a solid will have the strongest interactions between particles holding them together.

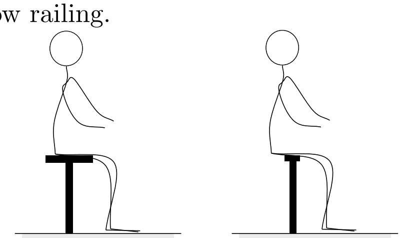
One way to determine whether something is solid or liquid is to push on its surface. If your finger easily breaks the surface, it is a liquid. If the surface can withstand the push of your finger without giving, then it is a solid. This does not help distinguish between a gas and a liquid. To distinguish a liquid from a gas, we could use the "compressibility test." Again this is a very qualitative definition: Push on all sides of a fluid so as to try and reduce its volume. You will find it much more difficult to do this for a liquid than for a gas.

This chapter will be broken into two major sections: static fluids and moving fluids. The discussion of moving fluids will be limited to incompressible liquids.

§ 2.1 Static Fluids

Pressure

We know that there is a difference between sitting on a bench and sitting on a narrow railing.



Even though the free body diagram is the same for these two situations, something is different. Our weight is spread over a larger area when we sit on a bench. The railing and bench both apply the same force on us, but the force is spread over a different area. The railing applies a greater force per area. If we sat on a nail that was sticking up from the bench, we would certainly notice that the force was spread over a very small area. Apparently, at times it is the force per area that is important.

Definition: **Pressure**
This ratio of the force over the area is called the *pressure.*

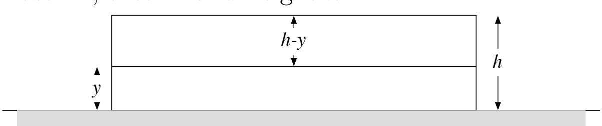
$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

$$P = \frac{F}{A}$$

The units of pressure are Newtons per square meter. This is commonly referred to as a Pascal (Pa).

EXAMPLE

Compute the pressure at a distance y from the bottom of a stack of paper of mass M, area A and height h.



$$P(y) = \frac{F_g}{A} = \frac{mg}{A} = \frac{M\frac{(h-y)}{h}g}{A} = \frac{M(h-y)g}{hA}$$

Note that the pressure difference can be written in terms of the change in height.

$$\Delta P = -\frac{M}{hA}g\Delta y$$

If we used a different sized piece of paper, would we expect the pressure to change? No, we would get the same result. Can we see this in the equation? Yes, because the mass increases in proportion to the area of the paper. So the ratio M/A is a constant. What is the constant M/Ah? Since Ah is the volume of the stack of paper and M is the mass of the stack of paper, M/Ah is the density! So we can write

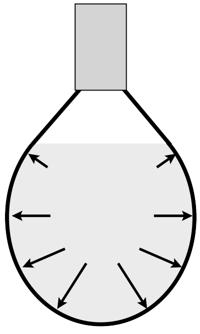
$$\Delta P = -\rho g \Delta y$$

The negative just tells us that the pressure decreases as we go up, and increases as we go down.

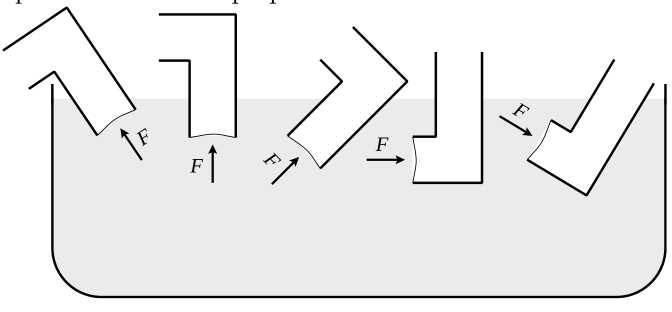
> *Definition:* **Density**
> The density of a substance is the ratio of its mass to its volume.
> $$\rho = \frac{m}{V}$$

Pressure in a Fluid

If you fill a plastic bag with water you can see that the fluid exerts a force on the bag. The force appears to always be perpendicular to the surface of the bag.



This is true with fluids: no matter what way you slice them they always push outward and perpendicular to the surface of the interface.



This is different from the pressure in the stack of paper. In the stack of paper the force was directed downward only. In fluids the pressure at a point can create a force in any direction.

Theorem: **Pressure**
It is a homework problem to show that the pressure in a fluid changes when you move up or down in the fluid. The result of the homework problem is that if you compare the pressure at two different positions in a fluid that have a difference in height of Δy, then the difference in pressure between these positions is

$$\Delta P = -\rho g \Delta y$$

where ρ is the density of the fluid.

For example the pressure at the bottom of a dam is greater than the pressure at the top of the dam.

EXAMPLE

What is the net force on a small observation window in a whale's holding tank. The window is a 10 cm square and is a distance of $h = 10$m below the surface? The pressure on one side of the window is the pressure of the water. The pressure on the other side is the pressure of the air. Use $\Delta P = -\rho g \Delta y$ twice: first between the surface and the water side and second between the surface and the air side. Assume the air pressure at the top of the tank is P_s and that the tank is open to the air. First compute the increase in pressure at a depth of 10m in the tank:

$$\begin{aligned} \Delta P &= -\rho g \Delta y \\ P_w - P_s &= -\rho_w g (y_w - y_s) \\ P_w - P_s &= -\rho_w g(-h) = 98000\text{Pa} \end{aligned}$$

Now compute the increase in the pressure of the air outside the window:

$$\begin{aligned} \Delta P &= -\rho g \Delta y \\ P_a - P_s &= -\rho_a g (y_a - y_s) \\ P_a - P_s &= -\rho_a g(-h) = 126\text{Pa} \end{aligned}$$

The force on the window is $F = F_w - F_a$ since the force of the air and the force of the water are in opposite directions. Thus

$$\begin{aligned} F = F_w - F_a = P_w A - P_a A &= (P_w - P_a)A \\ &= ((98000\text{Pa} + P_s) - (126\text{Pa} + P_s))A \\ &= (98000\text{Pa} - 126\text{Pa})A = 979\text{N} \end{aligned}$$

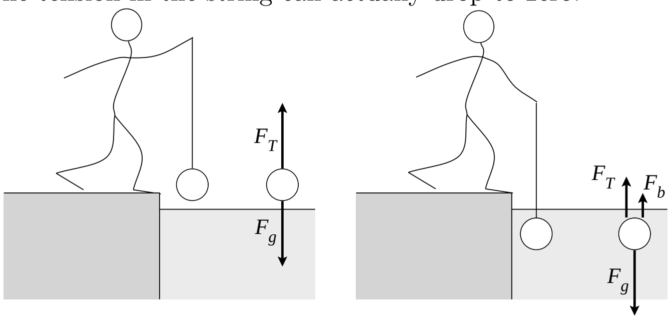
Notice that we would have induced an error of only about 0.1% if we had assumed the pressure in the air was the same at the surface and at the window.

⊙ Do This Now 2.1

A scuba diver in the ocean goes from the water's surface to a depth of 10m. By how much does the pressure on his body increase? If he is in a freshwater lake how much would the pressure increase?

Buoyancy

We know that things get "lighter" when they are in water. If we hang a weight from a string and then lower the weight into a pool of water the tension in the string decreases. If the object is able to float, then the tension in the string can actually drop to zero.



Since the gravitational force does not change when we put the weight in the water, we know that there must be another upward force that is supporting the weight, when we put the weight in the water. It must be that the water applies an upward force to objects in the water. This force is called the *buoyant force*.

By a simple observation we can determine the strength of the buoyant force. The observation is that the water surrounding the weight does the same thing to the weight as it did to the water that was where the weight is now. This implies that the buoyant force would be the same on any object that takes up the same space in the water. We know that this must be true since the water does not have intelligence, it simply pushes with the same strength on whatever happens to be next to it.

The reason that this observation will allow us to determine the strength of the buoyant force is that we can now figure the force on a chunk of water that has a volume equal to the volume of our object and know that the force on this chunk of water will be the same as the force on our object.

Consider then replacing the weight with a chunk of water of the same shape. Since it is the same as the rest of the water, this water will be in equilibrium. Thus we know that the net force on this must be zero. But there are only two forces acting on this chunk, gravity and the buoyant force, so these must be equal and opposite. We find then that the buoyant force is equal in magnitude to the weight of a chunk of water that is the same shape as our object:$F_b = m_w g$. This weight will only depend on the volume of the object not the detailed shape. We call the volume of the object that is underwater, V_d, the volume displaced. We can then write the mass of the chunk of water as $m_w = \rho_w V_d$. With this the buoyant force is $F_b = \rho_w V_d g$. This will work for any object submerged in any fluid.

Theorem: **Bouyant Force**
For an object in a fluid with density ρ_d the *bouyant force* on the object is

$$F_b = \rho_d V_d g,$$

where V_d is the mass of the fluid displaced by the object. Or equivalently

$$F_b = m_d g,$$

where m_d is the mass of the water displaced by the object.

EXAMPLE

Find the buoyant force on a cube of metal that is 10 cm on a side (a volume of 1 liter) and has a mass of 20 kg that 1) is fully submerged and 2) has only half of its volume submerged.
1) For the fully submerged cube, since one liter of water has a mass of 1 kg we can write:

$$F_b = m_d g = (1.0\text{kg})g = 9.8\text{N}$$

or

$$F_b = \rho_d V_d g = (1000\text{kg/m}^3)(0.10\text{m})^3 g = 9.8\text{N}$$

2) For the half-submerged cube, the volume of water displaced is 1/2 of the cube's volume, so

$$F_b = \rho_d V_d g = (1000\text{kg/m}^3)(1/2)(0.10\text{m})^3 g = 4.9\text{N}$$

EXAMPLE

An important measurement of a sailboard is its volume. This volume is usually given in liters. Suppose that a particular sailboard has a volume of 120 liters. And that the mass of the sailor, sail, and the sailboard is 80kg.
• What percentage of the board will be underwater when she stands on the board and holds up the sail? Since the buoyant force and

gravity are the only forces, we know that $F_b = mg$ But also $F_b = \rho_d V_d g$ so that $V_d = mg/\rho_d g = m/\rho_d$

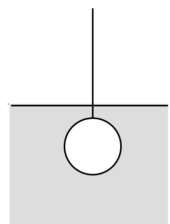
$$\frac{V_d}{V} = \frac{m/\rho_d}{V} = \frac{m}{\rho_d V} = \frac{80\text{kg}}{120\text{kg}} = 66.6\%$$

• The density of salt water is greater than the density of fresh water. If she stands on the board in saltwater ($\rho = 1025\text{kg/m}^3$) what percentage of the board will be underwater?

$$\frac{V_d}{V} = \frac{m}{\rho_d V} = \frac{80\text{kg}}{123\text{kg}} = 65.0\%$$

⊙ Do This Now 2.2

A weight with mass 10kg is fully submerged in water, being held up by a string:



The tension in the string is 92.9N. What is the bouyant force?

$B_F = F_g - T = 5.10\text{N}$

Example

In the previous example what is the density of the submerged weight? Use the bouyant force:

$$B_F = 5.10\text{N} = \rho_w V g,$$

where V is the weight's entire volume since it is fully submerged. The weight's volume is

$$V = \frac{B_F}{\rho_w g} = \frac{5.10\text{N}}{(1000\text{kg/m}^2)(9.8\text{m/s}^2)} = 5.20 \times 10 \times 10^{-4}\text{m}^3,$$

and the density is

$$\rho = \frac{m}{V} = \frac{10\text{kg}}{5.20 \times 10^{-4}\text{m}^3} = 19,200\text{kg/m}^3.$$

This is close to the density of pure gold.

▷ Problem 2.1

An elephant with bare feet and and a woman with spike-heeled shoes are walking over the ground. The elephant has a mass of about 1000kg

and its feet are nearly circular with a radius of about 10 cm. The woman has a mass of about 60kg and the heels of her shoes are nearly circular with a radius of about 0.4 cm.
(a) Compute the pressure that the elephant creates on the ground when she walks. (Note, the elephant has at least two feet on the ground at all times.)
(b) Compute the pressure that the woman creates on the ground when she walks.

▷ PROBLEM 2.2
A sheet of plywood is placed over an air mattress and then a 1000kg elephant stands on the plywood. The air mattress is 1.5 meters by 2.0 meters. You can create a pressure of about 0.5atm when you blow up the air mattress. Can you blow up the air mattress, while the elephant is standing on it?

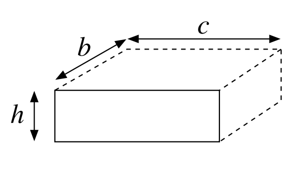
▷ PROBLEM 2.3
In the jungle you are trying to escape a lion. You decide to hide underwater in a muddy pond, using a hollow reed as a snorkel. If the maximum pressure difference your lungs can take before collapsing is 0.5atm, how deep can you go?

▷ PROBLEM 2.4
Consider a small rectangular blob of water that is a small part of all the water in a swimming pool full of water. Put an imaginary box around the blob of water as shown. Draw a free body diagram showing all the forces acting on this box of water. Be sure to include the force due to the pressure of the neighboring water.

Using Newton's second law show that the pressure difference in the fluid between the top and the bottom of the box is

$$\Delta P = P_{\text{top}} - P_{\text{bottom}} = -\rho g h$$

where ρ is the density of water.



This result shows us that if we move a vertical distance Δy in a fluid that the pressure will change:

$$\Delta P = -\rho g \Delta y$$

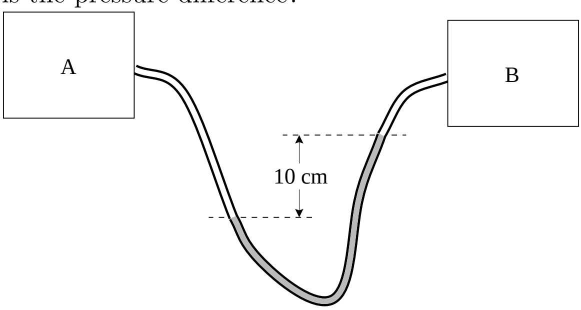
▷ PROBLEM 2.5
Hoover Dam, near Las Vegas, Nevada, is 730 feet high. What is the net force due to pressure on a one-square meter section of the dam at the bottom?

▷ PROBLEM 2.6
You go from the top of Mt Diablo (1173m) down to sea level.

(a) What is the change in pressure for this trip?
(b) Now you get in a submarine and go this same distance (1173m) below the surface of the ocean. What is the change in pressure for this second leg of your trip?

▷ PROBLEM 2.7
A professional scuba diver brings his lunch box to work on the sea floor at a depth of 200 meters. The lunch box is 10cm $\times$ 20cm $\times$ 30cm. Find the net force on each side of the box.

▷ PROBLEM 2.8
* A tube connecting two containers has a bit of fluid trapped in a low spot of the tube. The level of the fluid is 10cm higher on the side connected to container B. The density of the fluid is $\rho = 500\text{kg/m}^3$.
(a) Which container is at a greater pressure?
(b) What is the pressure difference?



§ 2.2 Moving Fluids

We have dealt with static situations for fluids. How do fluids move, whether water through a pipe or air across a wing?

Volume Rate of Flow

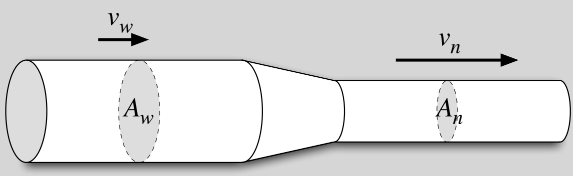
Suppose that a fluid is running through a pipe, at a velocity v. Commonly we would like to know at what rate the fluid is delivered. For example if you are filling a water tank you might want to know how long it will take to fill the tank. We know that the greater the velocity, the faster the tank will be filled. We also know that the greater the cross-sectional area of the pipe, the fast the tank will be filled. So it would seem that the quantity Av is the rate at which the pipe delivers fluid. Let us compute the quantity of fluid that runs from the end of the pipe in a time of Δt. In this time the fluid in the pipe will have moved a distance $\Delta x = v\Delta t$. So that a length of water Δx will have run from the pipe. This means that a volume $\Delta V = A\Delta x$ will have run from the pipe. Thus

$$\Delta V = A\Delta x = Av\Delta t \;\longrightarrow\; \frac{dV}{dt} = Av$$

This volume per time that flows past a point in the pipe is called the volume rate of flow.

Theorem: **Equation of Continuity**

Notice that if the pipe narrows, the volume rate of flow can not change, since the flow must go someplace.



(We are assuming here that the fluid does not compress). Thus if the area of the wide section is A_w and the velocity at the wide section is v_w and similarly for the narrow section we must have the following.

$$A_w v_w = A_n v_n$$

EXAMPLE

The pipe going into a house 3/4 inches (1.9cm) in diameter (inside dimension). A "T" in the 3/4 inch line takes a 1/2 inch (1.3cm) diameter pipe up to the kitchen faucet. If the only water flowing in the house is through the kitchen faucet and the water enters the house at a rate of 1.5 m/s, at what speed does the water flow out of the sink?

Use the equation of continuity:

$$A_w v_w = A_n v_n,$$

where the narrow section of pipe is to the faucet and the wide section is into the house:

$$\longrightarrow \quad v_n = \frac{A_w}{A_n} v_w = \frac{1.9^2}{1.3^2}(1.5\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 3.2\,\tfrac{\mathrm{m}}{\mathrm{s}}.$$

⊙ DO THIS NOW 2.3

If you use the faucet in the previous example to fill a 5 gallon (19 liter) bucket, how long will it take?

$t_{fill} = 45s$

Bernoulli's Equation

Demo

Attractive Paper: If you blow between two sheets of paper that are hanging down you would expect the pressure of the air between the sheets to be greater (since we are blowing) than the pressure outside the sheets. Thus we expect the sheets to be pushed apart when we blow. Our intuition in this case is incorrect. The sheets are drawn together.

So much for that line of reasoning. It went wrong at some point. The error was in assuming that the air was at a high pressure between the sheets. In fact by the motion of the paper we see that the pressure is less between the sheets.

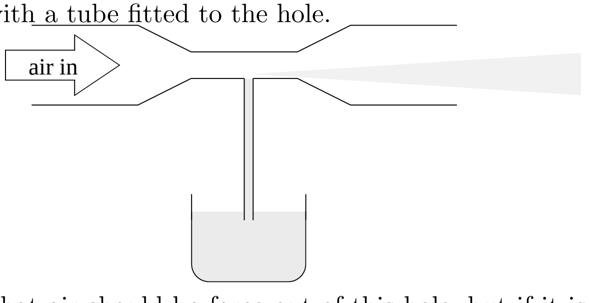
To clarify this reconsider the pipe that narrows in the previous theorem. We know that the velocity is greater in the narrow section. This means that as the fluid passed into this narrow section it was accelerating. Thus there must have been a net force toward the narrow section. This means that the pressure of the water behind must have been greater than the pressure of the water ahead. So it must be the case that the pressure is lower in the narrow section. This is counterintuitive. So, **be careful**.

Demo

The Balancing Ball: If you place a ball in the air stream coming from a blower we find that the ball is drawn toward the center of the stream. The ball is drawn toward the fast moving air in the center of the stream. This makes sense only if the pressure is less where the air is moving faster. (This would be consistent with the previous demonstration).

Demo

Venturi Tube: Let us make one final test of this idea that fast moving air is at a lower pressure. We will force air through a pipe with a constriction in the middle. This middle section has a small whole in the side with a tube fitted to the hole.

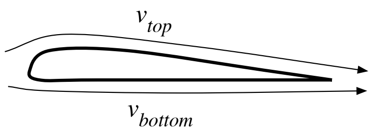


It seems that air should be force out of this hole, but if it is true that fast moving air is at a lower pressure then the middle section should be at a lower pressure and thus will draw air into the hole. To test this we can put the tube in a container of water and see if it draw water in or blows bubbles.

This system is used to spray perfume: a small squeezable bulb is used to force air through the narrow section, and the perfume is drawn up into the air stream and spewed out the end. The bottle using this technique to disperse perfume is called an "atomizer."

A engine carburetor also works by this principle. The air flowing into the engine is drawn through a narrow section. Gasoline is lead to this section by a small tube. The low pressure in this section draw the gasoline from the tube into the airflow which is headed toward the engine.

An wing also works by the same principle. The air going over the top of the wing goes faster and thus the pressure is lower on this side. This causes a net force on the wing.



Theorem: **Bernoulli's equation**
By using the work energy theorem we can find the following result, which allows us to compute the pressure difference between two points in a moving fluid.

$$\Delta[P + \tfrac{1}{2}\rho v^2 + \rho g y] = 0$$

or

$$\Delta P + \Delta[\tfrac{1}{2}\rho v^2] + \Delta[\rho g y] = 0$$

Thus the quantity $[P + \frac{1}{2}\rho v^2 + \rho g y]$ is the same at all points in fluid. (It is assumed that the fluid is flowing without turbulence or friction).

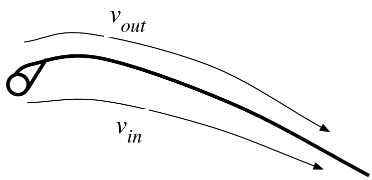
There are two special cases of Bernoulli's equation that are worth mentioning.

First notice that if the velocity is the same at both points, then $\Delta v^2 = 0$ and thus $\Delta P + 0 + \rho g \Delta y = 0 \longrightarrow \Delta P = -\rho g \Delta y$. This is the equation we have been using to find the pressure difference due to a difference in elevation. Thus we can just remember Bernoulli's equation and forget the more limited one. You can then use those freed-up brain cells for something else, like your mother's shoe size.

Second notice that in air the term $\rho g \Delta y$ gives a minor pressure difference, unless Δy is very large. Thus we can usually ignored this term.

EXAMPLE

Suppose a sail is designed so that the air moving over the inside of the sail travels at a speed that is 90% of the speed of the air moving over the outside of the sail: $v_{\text{in}} = 0.90 v_{\text{out}}$. The area of the sail is 4.5m^2.



How fast will the air need to go over the outside of the sail in order to produce a force of 200 Newtons?

$$
\begin{aligned}
F &= F_{\text{in}} - F_{\text{out}} \\
&= P_{\text{in}}A - P_{\text{out}}A = \Delta P A = -\Delta[\tfrac{1}{2}\rho v^2]A \\
&= -[\tfrac{1}{2}\rho v_{\text{in}}^2 - \tfrac{1}{2}\rho v_{\text{out}}^2]A \\
&= -[\tfrac{1}{2}\rho(0.90 v_{\text{out}})^2 - \tfrac{1}{2}\rho v_{\text{out}}^2]A \\
&= -\tfrac{1}{2}\rho[(0.90)^2 - 1]A v_{\text{out}}^2 \\
&= (0.19)\tfrac{1}{2}\rho A v_{\text{out}}^2 \\
\longrightarrow \quad v_{\text{out}} &= \sqrt{\frac{F}{(0.19)\frac{1}{2}\rho A}} = 19\,\tfrac{\text{m}}{\text{s}}
\end{aligned}
$$

Notice that the term $\rho g \Delta y$ would have contributed a force of $F = A\rho g \Delta y \approx 0.5$N. This is small compared with the force of 200 N.

▷ PROBLEM 2.9

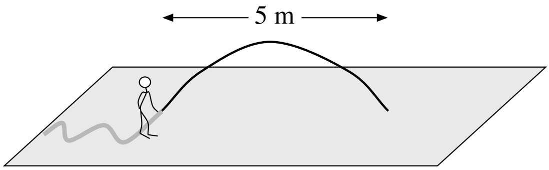
You wade across a stream and find that the speed of the water is $0.62\frac{\text{m}}{\text{s}}$ at this part of the stream. The stream is 4.2 meters wide and 0.85 meters deep where you walked across.

(a) What is the volume rate of flow of this stream?

(b) You now move down to a narrow point in the stream. Here the stream is only 1.2 meters wide and moving at a speed of $2.7\frac{\text{m}}{\text{s}}$. How deep is the stream at this narrow point?

▷ PROBLEM 2.10

You are watering a lawn. When you squirt the hose at a 45° angle as shown, the water lands about five meters from you.



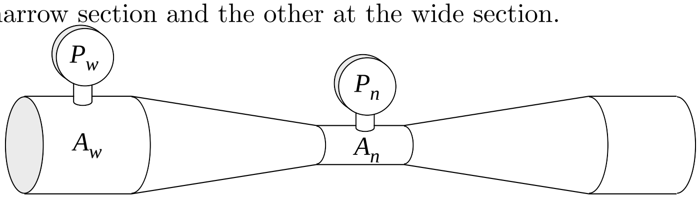
(a) Estimate the velocity of the water as it leaves the nozzle?
(b) The opening in the nozzle is a circular hole with a diameter of 4 mm. What is the volume rate of flow from the nozzle?
(c) You proceed to water a lawn with this hose. The lawn is a square that is 10 meters on a side. How long will it take to give the lawn an equivalent of 2.0 cm of rain?

▷ Problem 2.11

The wings of an airplane give a lift of 10^4N and they have an area of 10m^2. The air is moving at a speed of $300\,\frac{\text{m}}{\text{s}}$ under to bottom of the wing. What is the speed of the air moving over the top of the wing?

▷ Problem 2.12

A tube with a fluid running through it has a wide section with cross-sectional area A_w and a narrow section with cross-sectional area A_n. There are two pressure sensors attached to the tube, one is placed at the narrow section and the other at the wide section.



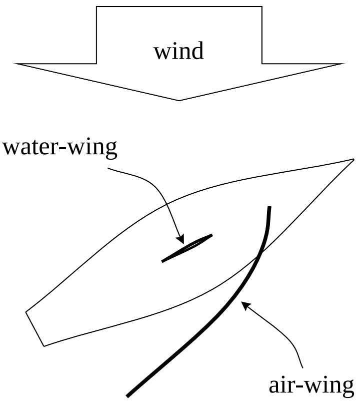
From the pressure readings P_w and P_n you can determine the volume rate of flow in the tube.
(a) Show that in general, the volume rate of flow in this system is given by the equation below.

$$\frac{dV}{dt} = \sqrt{\frac{2(P_w - P_n)}{\rho\left(\frac{1}{A_n^2} - \frac{1}{A_w^2}\right)}}$$

(b) Suppose the fluid is water, the radius of the wide section is 2.0cm and the radius of the narrow section is 1.0cm and the pressures are $P_w = 1.2 \times 10^5$Pa and $P_n = 1.1 \times 10^5$Pa. What is the volume rate of flow?

▷ Problem 2.13

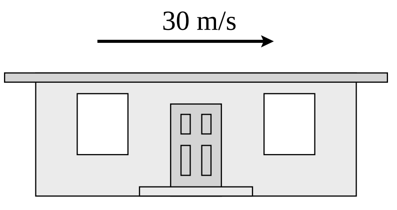
Sailboats can sail upwind. The reason that a sailboat can move upwind is that a sailboat has two "wings". The sail is a wing that extends upward from the boat into the air. What is not obvious is that there is also a wing that extends downward into the water from the bottom of the boat.



Show with a force diagram that the lift from these two wings can give a net force in the forward direction even if this forward direction is upwind.

▷ PROBLEM 2.14

Suppose that you have a house with a flat roof as shown and that there is a strong wind blowing over the house at a speed of $30\,\frac{\text{m}}{\text{s}}$. The roof is 8 meters by 13 meters.

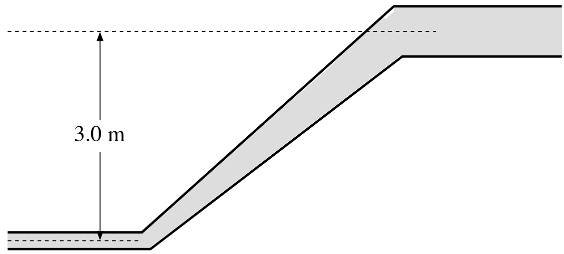


(a) What is the net force on the roof due to the pressure difference between the inside and outside of the house?

(b) Is this force upward or downward?

▷ PROBLEM 2.15

A pipe carrying gasoline goes up a short vertical rise of 3.0 meters while at the same time the radius of the pipe doubles. The velocity of the gasoline is $8.0\,\frac{\text{m}}{\text{s}}$ in the smaller section of pipe. What is the change in the pressure of the gasoline as it goes from the smaller pipe to the larger pipe?



▷ Problem 2.16

Consider a wing. Let the velocity of the wing in the fluid be v and assume that the change in velocities of the fluid on the fast and slow sides of the wing are proportional to the velocity. This assumption seems justified since the geometry of the wing should determine the relative velocity shift $\frac{\delta v}{v} = \epsilon$ and so this ratio is expected to be independent of the velocity. With this assumption we are lead to the expression of the velocities on the fast and slow sides of the wing as

$$v_{\text{fast}} = v + \delta v = v\left(1 + \frac{\delta v}{v}\right) = v(1+\epsilon)$$

$$v_{\text{slow}} = v - \delta v = v\left(1 - \frac{\delta v}{v}\right) = v(1-\epsilon)$$

Now show that if we ignore the gravitational potential energy that the lift on the wing is

$$F_{\text{lift}} = \tfrac{1}{2}\rho v^2 A C_L$$

where $C_L = 4\epsilon$. The constant C_L is called the *lift coefficient*.

§ 2.3 Summary

Definitions

- Density

$$\rho = \frac{m}{V}$$

- Pressure

$$P = \frac{F}{A}$$

Theorems

- Volume rate of flow

$$\frac{dV}{dt} = Av$$

- Equation of continuity for incompressible fluid

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}$$

- Bernoulli's Equation

$$\Delta[P + \tfrac{1}{2}\rho v^2 + \rho g y] = 0$$

3 Holthuijsen Ch 1 & 2

§ 3.1 Problems Ch 1

▷ Problem 3.1
In Holthuijsen chapter 1 the author describes different scales. Scale (2) is the subject addressed by most of the book. The author justifies why the detailed analysis used for scale (1) is not used in scale (2), with reasons **a** through **d**. The predictive power of the spectral analysis used in scale (2) depends in large part on the extent to which the linear approximation of the theory is justified. The linear theory implies that sinusoidal component waves exist independently on the surface, not effecting each other. There are cases in which the full non-linear theory allows one wave to “rob” energy from other waves. This collecting of energy into a single wave leads to waves of much larger amplitude than is predicted by the linear theory. The existence of these unexpectedly large waves would make unreasonable which of the reasons **a** through **d**?

§ 3.2 Observation Techniques

- Buoys: accelerometer on a buoy.
- Wave Poles: resistance or capacitance of sea water. Requires fixed platform in the sea.
- Tide Gauges: pressure transducer on the sea floor.
- Lidar Altimeter: distance to sea surface from a platform of known position. Platform could be airborne or on a satellite.
- Radar Altimeter: distance to sea surface from a platform of known position. Platform could be airborne or on a satellite.
- Imaging SAR: distance to sea surface from a moving platform, observes many places at once. Because of side-looking antenna accuracy is limited. SAR is Synthetic Aperture Radar, a technique of faking a very large antenna to be able to focus on a small area.
- SAR Altimeter: hybrid between Radar Altimeter and imagining SAR.

§ 3.3 Problems Ch 2

▷ Problem 3.2
What are the two most common techniques to measure waves at sea?

▷ PROBLEM 3.3
What are the two most common techniques to obtain directional wave information based on these techniques?

▷ PROBLEM 3.4
Is it possible to measure the surface profile of individual waves from satellites?

▷ PROBLEM 3.5
On the class webpage is a data file `buoy.mat` or `buoy.txt` which contains buoy data from a buoy off the California coast Point Reyes CA. The data is sampled at a rate of 1.28 samples per second. Compute the Fourier transform and plot the power versus frequency.

▷ PROBLEM 3.6
The antenna gain of a radar altimeter is

$$G = 2^{-(x^2+y^2)/(h\theta)^2} = e^{-\ln(2)(x^2+y^2)/(h\theta)^2}$$

with the antenna beam width at half power of $\theta = 0.01$rad, and the satellite altitude of $h = 717$km, and (x, y) is the location on the sea surface, with $(0, 0)$ directly below the satellite. If the height of the sea at location (x, y) is z then the distance between the point and the satellite is

$$r = \sqrt{(h - z)^2 + x^2 + y^2}.$$

Simulate the return power from the sea surface versus time for the cases below. If you get stuck look at the hints section.
(a) A perfectly calm sea, thus the sea surface height is $z = 0$
(b) There is a perfect swell of height H and wavelength L.

$$z = \frac{H}{2}\cos(k_x x + k_y y)$$

with $\vec{k}$ the vector wave number for the swell, that is $\vec{k}$ it is a vector of magnitude $\frac{2\pi}{L}$ who's direction is the direction the swell is moving.
(c) The sea surface height z is a random variable uniformly distributed from $-H/2$ to $H/2$
(d) The sea surface height z is a random variable with a gaussian distribution with standard deviation $H/4$.
(e) The sea surface height z is the superposition of a perfect swell of height H_s and a gaussian random sea with standard deviation $H_w/4$.

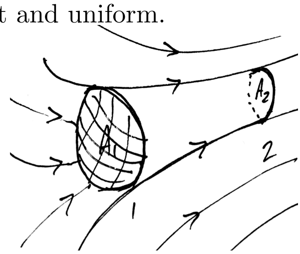
4 Hydrodynamics

§ 4.1 Representation of fluid flow

At each fixed point at each time the fluid will move with a particular velocity. We will write this velocity as the time dependent vector field $\vec{u}(x, y, z, t)$. We denote the density of the fluid as a scalar field $\rho(x, y, z, t)$. In our case the fluid we are going to study is the sea, and we are particularly interested in the surface of the sea, that is the interface between the water and the air. This interface will be described by the scalar field $\eta(x, y, t)$ which gives the displacement in the vertical direction of the surface from the equilibrium height at position (x, y) at time t.

§ 4.2 Bernoulli's Equation

We will begin by considering the simpler case of a constant flow in which case we know that $\frac{\partial \vec{u}}{\partial t} = 0$. We will also suppose that the density of the fluid is constant and uniform.



Consider a small surface A_1 who's normal is everywhere parallel to $\vec{u}$. The mass rate or flow through this area is

$$\frac{dm}{dt} = \frac{d[\rho\mathcal{V}]}{dt} = \rho\frac{d\mathcal{V}}{dt} = \rho A_1 u_1$$

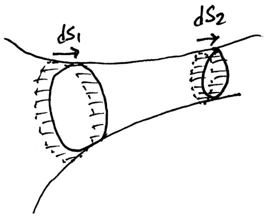
Consider now another surface A_2 bound by the same stream lines as A_1, but further down the flow. The mass rate or flow though A_2 is

$$\frac{dm}{dt} = \rho A_2 u_2$$

But no mass flows across the stream lines so since mass is not building up or leaving the volume enclosed by the stream lines and the two areas, we see that the mass flow rate the two ends must be the same. Thus we find that

$$A_1 u_1 = A_2 u_2 = \text{constant}$$

and we call this constant the volume rate of flow. Let us suppose that the volume rate of flow through this particular stream line bounded region is R.



Now let's follow the mass that is between A_1 and A_2 for a short time dt. A volume of water $R\ dt$ will come out of the A_2 and the volume that $R\ dt$ that was up against A_1 will have moved down the stream a distance ds_1. We want to compute the change in the mechanical energy of this section of water. While all of the water has moved, the energy difference is the same as if we just moved the water from the left $R\ dt$ to the right $R\ dt$. Thus the change in energy of the total mass of water is the energy difference of these two slugs of water: $dE = dE_2 - dE_1$.

$$dE_1 = \tfrac{1}{2} dm_1 v_1^2 + dm_1 g z_1 = \tfrac{1}{2}\rho R\ dt\ v_1^2 + \rho R\ dt\ g z_1$$

likewise

$$dE_2 = \tfrac{1}{2}\rho R\ dt\ v_2^2 + \rho R\ dt\ g z_2$$

and so

$$dE = \tfrac{1}{2}\rho R\ dt\ (v_2^2 - v_1^2) + \rho R\ dt\ g(z_2 - z_1)$$

By the work energy theorem this change in energy must be equal to the net work done on the mass of water. At A_1 the pressure does an amount of work on the water of

$$dW_1 = \vec{F}_1 \cdot \vec{ds}_1 = P_1 A_1 ds_1 = P_1 A_1 u_1 dt = P_1 R\ dt$$

while at A_2 the pressure does an amount of work

$$dW_2 = \vec{F}_2 \cdot \vec{ds}_2 = -P_2 R\ dt$$

since the displacement and the pressure force are in opposite directions the dot product is negative at A_2. The net work is thus

$$dW = dW_2 + dW_1 = -(P_2 - P_1) R\ dt$$

Since we know that $dE = dW$ we can say that

$$\tfrac{1}{2}\rho R\ dt\ (v_2^2 - v_1^2) + \rho R\ dt\ g(z_2 - v_1) = -(P_2 - P_1) R\ dt$$

or

$$\tfrac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(z_2 - z_1) + (P_2 - P_1) = 0$$

or

$$\tfrac{1}{2}\rho v^2 + \rho g z + P = \text{constant}$$

§ 4.3 A bug in water

Suppose that you have a bug that is a fluid and moving with the fluid. Also suppose that you have a temperture field for the fluid, $T(x, y, z, t)$ which gives the temperature of the fluid at position (x, y, z) at time t. Now we would like to know the position $\vec{r}(t)$ and temperature $T_b(t)$ of the bug as it moves along with the fluid. We want to write these things out in terms of the velocity field $\vec{u}(x, y, z, t)$ of the fluid.

Let the position of the bug be $\vec{r}(t) = r_x(t)\hat{x} + r_y(t)\hat{y} + r_z(t)\hat{z}$. Then we note that the velocity of the bug is

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \vec{u}(r_x(t), r_y(t), r_z(t), t)$$

and so

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{u}(r_x(\tau), r_y(\tau), r_z(\tau), \tau) d\tau$$

The temperature of the bug T_b can be computed from the temperature field $T(x, y, z, t)$

$$T_b(t) = T(r_x(t), r_y(t), r_z(t), t)$$

So that

$$\begin{aligned}
\frac{dT_b}{dt} &= \frac{d}{dt} T(r_x(t), r_y(t), r_z(t), t) \\
&= \frac{\partial T}{\partial t}\frac{dt}{dt} + \frac{\partial T}{\partial x}\frac{dr_x}{dt} + \frac{\partial T}{\partial y}\frac{dr_y}{dt} + \frac{\partial T}{\partial z}\frac{dr_z}{dt} \\
&= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}u_x + \frac{\partial T}{\partial y}u_y + \frac{\partial T}{\partial z}u_z \\
&= \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z} \\
&= \frac{\partial T}{\partial t} + \left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right) T \\
&= \frac{\partial T}{\partial t} + (\vec{u} \cdot \nabla)T
\end{aligned}$$

The object $(\vec{u} \cdot \nabla)$ is called the directional derivative and is the rate of change in the direction $\vec{u}$.

We can find the acceleration of the bug in a similar way.

$$\begin{aligned}\vec{a}(t) &= \frac{d\vec{v}}{dt} = \frac{d}{dt}\vec{u}(r_x(t), r_y(t), r_z(t), t) \\ &= \frac{\partial \vec{u}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{u}}{\partial x}\frac{dr_x}{dt} + \frac{\partial \vec{u}}{\partial y}\frac{dr_y}{dt} + \frac{\partial \vec{u}}{\partial z}\frac{dr_z}{dt} \\ &= \frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{u}}{\partial x}u_x + \frac{\partial \vec{u}}{\partial y}u_y + \frac{\partial \vec{u}}{\partial z}u_z \\ &= \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\end{aligned}$$

Theorem: **Acceleration of a particle following a flow**
If the velocity field in a fluid is $\vec{u}$ then the acceleration of a particle moving with the field is given by

$$\vec{a}(t) = \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}$$

There is a vector calculus identity that says

$$\tfrac{1}{2}\nabla(\vec{u}\cdot\vec{u}) = (\vec{u}\cdot\nabla)\vec{u} + \vec{u}\times(\nabla\times\vec{u})$$

With this we can write

Theorem: **Acceleration of a particle following a flow: alt**
If the velocity field in a fluid is $\vec{u}$ then the acceleration of a particle moving with the field is given by

$$\vec{a}(t) = \frac{\partial \vec{u}}{\partial t} + \tfrac{1}{2}\nabla(u^2) - \vec{u}\times(\nabla\times\vec{u})$$

▷ PROBLEM 4.1
Consider a bucket of water that has been set in motion, circling around counter clockwise inside the bucket with a constant angular velocity ω. There is a bug in the water. Working in rectangular coordinates do the following.
(a) Show that the velocity field is $\vec{u} = -\omega y\ \hat{x} + \omega x\ \hat{y}$.
(b) Show that $\vec{u}\cdot\nabla = -\omega y\frac{\partial}{\partial x} + \omega x\frac{\partial}{\partial y}$.
(c) Compute $(\vec{u}\cdot\nabla)\vec{u}$.
(d) Compute the acceleration of the bug using $\vec{a}(t) = \frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}$
(e) Compare this acceleration with what you know about going in circles from first semester physics.
(f) Show that $\nabla\times\vec{u} = 2\omega\hat{z}$.
(g) Show that $\frac{1}{2}\nabla(\vec{u}\cdot\vec{u}) = (\vec{u}\cdot\nabla)\vec{u} + \vec{u}\times(\nabla\times\vec{u})$ by computing both sides of the equation. Note that you have already done most of this.

§ 4.4 Identities from vector calculus

We will need some results from your vector calculus class.

We consider a closed surface S bounding the area A, in which there is a scalar field ψ and vector fields $\vec{u}$ and $\vec{v}$. In the expression below $\vec{dA}$ is a surface element in the direction of the outward normal.

Divergence theorem:

$$\int_V \nabla \cdot \vec{u}\, dV = \oint_S \vec{u} \cdot \vec{dA}$$

Noname theorem:

$$\int_V \nabla\psi\, dV = \oint_S \psi\, \vec{dA}$$

§ 4.5 Einstein notation

We use the notation

$$\begin{aligned} \hat{e}_1 &= \hat{x} \\ \hat{e}_2 &= \hat{y} \\ \hat{e}_3 &= \hat{z} \end{aligned}$$

for the coordinate unit vectors. We write

$$\vec{u} = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3 = u_i\hat{e}_i$$

where there is an implicit sum over i, that is $u_i\hat{e}_i$ a shorthand for

$$u_i\hat{e}_i \equiv \sum_{i=1}^{3} u_i\hat{e}_i.$$

Using the same notation we can write out other things as well.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_i v_i \\ \nabla \cdot \vec{v} &= \partial_i v_i = \frac{\partial v_i}{\partial x_i} \\ \nabla\psi &= \partial_i \psi\ \hat{e}_i = \frac{\partial \psi}{\partial x_i}\hat{e}_i \end{aligned}$$

Definition: **Levi-Civita symbol**
Define the Levi-Civita symbol ε_{ijk} by the following rules.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1) \text{ or } (2,1,3) \text{ or } (1,3,2) \\ 0 & \text{otherwise} \end{cases}$$

Theorem: **Levi-Civita symbol patterns**
The following are always true for any i, j, and k.

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$$

and

$$\varepsilon_{ikj} = -\varepsilon_{ijk} \qquad \text{AND} \qquad \varepsilon_{kji} = -\varepsilon_{ijk} \qquad \text{AND} \qquad \varepsilon_{jik} = -\varepsilon_{ijk}$$

Using the Einstein notation and the Levi-Civita symbol we can write the cross product in a simple form. In some sense it has 27 terms but 21 of these terms are zero, adding these zero terms allows us to write the cross product in a uniform way.

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_2 v_3 - u_3 v_2)\,\hat{e}_1 - (u_1 v_3 - u_3 v_1)\,\hat{e}_2 + (u_1 v_2 - u_2 v_1)\,\hat{e}_3 \\
&= u_2 v_3 \hat{e}_1 - u_3 v_2 \hat{e}_1 - u_1 v_3 \hat{e}_2 + u_3 v_1 \hat{e}_2 + u_1 v_2 \hat{e}_3 - u_2 v_1 \hat{e}_3 \\
&= \varepsilon_{231} u_2 v_3 \hat{e}_1 + \varepsilon_{321} u_3 v_2 \hat{e}_1 + \varepsilon_{132} u_1 v_3 \hat{e}_2 \\
&\quad + \varepsilon_{312} u_3 v_1 \hat{e}_2 + \varepsilon_{123} u_1 v_2 \hat{e}_3 + \varepsilon_{213} u_2 v_1 \hat{e}_3 \\
&= \varepsilon_{231} u_2 v_3 \hat{e}_1 + \varepsilon_{321} u_3 v_2 \hat{e}_1 + \varepsilon_{132} u_1 v_3 \hat{e}_2 \\
&\quad + \varepsilon_{312} u_3 v_1 \hat{e}_2 + \varepsilon_{123} u_1 v_2 \hat{e}_3 + \varepsilon_{213} u_2 v_1 \hat{e}_3 \\
&\quad + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\
&\quad + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\
&= \varepsilon_{ijk} u_i v_j \hat{e}_k
\end{aligned}$$

Thus the k-th component of $\vec{u} \times \vec{v}$ is $[\vec{u} \times \vec{v}]_k = \varepsilon_{ijk} u_i v_j$ where there is no implicit sum over k because there is only one k in the expression.

In the same way we can write.

$$\nabla \times \vec{u} = \varepsilon_{ijk} \partial_i u_j \hat{e}_k$$

▷ Problem 4.2
Prove that $\nabla \cdot [\rho \vec{u}] = \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u}$.

▷ Problem 4.3
Show that $\vec{u} \times (\nabla \times \vec{u}) = \frac{1}{2} \nabla (\vec{u} \cdot \vec{u}) - (\vec{u} \cdot \nabla) \vec{u}$.

▷ Problem 4.4
Prove that $\nabla \times (\vec{u} \times \vec{v}) = \vec{u}(\nabla \cdot \vec{v}) - \vec{v}(\nabla \cdot \vec{u}) + (\vec{v} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\vec{v}$

§ 4.6 Equation of Continuity

The volume rate of flow going though an infinitesimal area $d\vec{A}$ is

$$\frac{dV}{dt} = \vec{u} \cdot d\vec{A}$$

Thus the rate of mass flowing through the area is

$$\frac{dm}{dt} = \rho \frac{dV}{dt} = \rho \vec{u} \cdot d\vec{A}$$

Consider an arbitrary closed surface S in the fluid flow. The rate of change of mass in the volume is

$$\frac{dm}{dt} = -\oint_S \rho\vec{u}\cdot d\vec{A}$$

where the direction of $d\vec{A}$ is taken to be the outward normal. But by the divergence theorem this is also

$$\frac{dm}{dt} = -\oint_S \rho\vec{u}\cdot d\vec{A} = -\int_V \nabla\cdot[\rho\vec{u}]\ dV$$

where the integral is over the volume V contained inside the surface S. Now we also know that the total mass in the volume is

$$m = \int_V \rho\ dV \quad\longrightarrow\quad \frac{dm}{dt} = \int_V \frac{\partial\rho}{\partial t}dV$$

and so

$$\int_V \frac{\partial\rho}{\partial t}dV = -\int_V \nabla\cdot[\rho\vec{u}]\ dV$$

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot[\rho\vec{u}]\right]\ dV = 0$$

But the surface S is arbitrary so the only way for this integral to always be zero is for the integrand to be zero!

$$\frac{\partial\rho}{\partial t} + \nabla\cdot[\rho\vec{u}] = 0$$

There is a "product rule" for the divergence, like there is for the regular derivative. The divergence of a product of a scalar field and a vector field can be written as $\nabla\cdot[\rho\vec{u}] = \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u}$ so the above can also be written as

$$\frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0$$

Theorem: **Equation of continuity**
The three following expressions are equivalent.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot[\rho\vec{u}] = 0$$

$$\frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0$$

$$\frac{d\rho_b}{dt} + \rho\nabla\cdot\vec{u} = 0$$

where ρ_b is the density of the water around the bug that is flowing with the water.

Notice that if the fluid is incompressible that the density of the water around the bug does not change, $\frac{d\rho_b}{dt} = 0$, and thus $\nabla\cdot\vec{u} = 0$.

Theorem: **Equation of continuity for incompressible fluid**

$$\nabla \cdot \vec{u} = 0$$

▷ PROBLEM 4.5
Consider the following velocity fields. Do they satisfy the equation of continuity for an incompressible fluid?
(a) $\vec{u} = -\omega y \; \hat{x} + \omega x \; \hat{y}$.
(b) $\vec{u} = \omega x \; \hat{x} - \omega y \; \hat{y}$.
(c) $\vec{u} = \alpha(y^2 - x^2) \; \hat{x} + 2\alpha xy \; \hat{y}$.
(d) $\vec{u} = \sinh(\kappa x) \cosh(\kappa y) \; \hat{x} - \cosh(\kappa x) \sinh(\kappa y) \; \hat{y}$.

§ 4.7 Force caused by pressure

We saw in the previous chapter that due to energy considerations that the pressure was related to changes in velocity in the fluid (Bernoulli's equation). It is also possible to look at this relationship as the pressure causing a force on the water which leads to an acceleration of the water. Let us start this investigation by looking at the force caused by pressure.

We consider a volume V of water contained by a closed surface S. The net pressure force on the volume of water will be result of the inward pressure on the surface

$$\vec{F}_P = \oint_S P \; (-\hat{n}) dA = - \oint_S P \; d\vec{A}$$

where the direction of the vector $d\vec{A}$ is the outward normal $\hat{n}$ of the surface. But we can express this surface integral as a volume integral of the gradient of the pressure.

$$\vec{F}_P = - \int_V \nabla P \; dV$$

Thus the pressure force per mass is

$$\vec{p} = \frac{F_P}{m} = -\frac{\int_V \nabla P \; dV}{\int_V \rho \; dV}$$

If we take the limit as the volume becomes small the gradient and density become uniform over the volume and thus

$$\vec{p} = \lim_{V \to 0} \frac{F_P}{m} = - \lim_{V \to 0} \frac{\int_V \nabla P \; dV}{\int_V \rho \; dV} = - \lim_{V \to 0} \frac{\nabla P \; V}{\rho \; V} = -\frac{\nabla P}{\rho}$$

§ 4.8 Force caused by gravity

In the same way we can get the force caused by gravity per mass. The net gravitational force on a volume V of water is

$$\vec{F}_G = \int_V (-g\hat{z})\ \rho\ dV = -g\hat{z} \int_V \rho\ dV$$

where g is the magnitude of the local gravitational field. The gravitational force per mass is then

$$\vec{g} = \frac{\vec{F}_G}{m} = \frac{-g\hat{z} \int_V \rho\ dV}{\int_V \rho\ dV} = -g\hat{z}$$

which is just the gravitational field.

§ 4.9 From Newton's Second Law

Now we can write out Newton's second law for a small volume V of water.

$$\begin{aligned} m\vec{a} &= \sum \vec{F} \\ \longrightarrow\ \vec{a} &= \sum \frac{\vec{F}}{m} \\ \vec{a} &= \frac{\vec{F}_P}{m} + \frac{\vec{F}_G}{m} \\ \vec{a} &= \vec{p} + \vec{g} \end{aligned}$$

$$\frac{\partial \vec{u}}{\partial t} + \tfrac{1}{2}\nabla(u^2) - \vec{u} \times (\nabla \times \vec{u}) = -\frac{\nabla P}{\rho} - g\hat{z}$$

Theorem: **Equation of motion for a fluid**

$$\frac{\partial \vec{u}}{\partial t} + \tfrac{1}{2}\nabla(u^2) - \vec{u} \times (\nabla \times \vec{u}) + \frac{\nabla P}{\rho} + g\hat{z} = 0$$

In a flow with conservative forces the fluid is irrotational.

$$\nabla \times \vec{u} = 0$$

and thus can be expressed as the gradient of a scalar potential field

$$\vec{u} = \nabla \phi$$

We call ϕ the *velocity potential* field. Putting this into our equation of motion we get.

$$\frac{\partial \nabla \phi}{\partial t} + \tfrac{1}{2}\nabla\left(|\nabla\phi|^2\right) + \frac{\nabla P}{\rho} + g\hat{z} = 0$$

or

$$\nabla\left[\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + \frac{P}{\rho} + gz\right] = 0$$

which implies that

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + \frac{P}{\rho} + gz = \text{constant}$$

> *Theorem:* **Equation of motion for irrotational fluid**
> With $\vec{u} = \nabla\phi$
>
> $$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + \frac{P}{\rho} + gz = \text{constant}$$

For an incompressible fluid we know that the divergence of the velocity field is zero. Thus for an irrotational incompressible fluid we can say that

$$\nabla\cdot\vec{u} = \nabla\cdot\nabla\phi = \nabla^2\phi = 0$$

So the velocity potential satisfies Laplace's equation.

> *Theorem:* **Equation of motion for irrotational incompressible fluid**
> With $\vec{u} = \nabla\phi$
>
> $$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + \frac{P}{\rho} + gz = \text{constant}$$
>
> and
>
> $$\nabla^2\phi = 0$$

▷ Problem 4.6
Let us return to the bucket of swirling water that we keep talking about. Recall that we already calculated that

$$\vec{a} = \frac{\partial \vec{u}}{\partial t} + \tfrac{1}{2}\nabla(u^2) - \vec{u}\times(\nabla\times\vec{u}) = -\omega^2\vec{r}.$$

(a) Use the equation of motion for a fluid to find the pressure in the fluid at an elevation z and at a distance r from the center.
(b) Let the height of the upper surface of the water be the function $\eta(r)$. Presume that the pressure of the water is zero at the surface to find $\eta(r)$.

▷ Problem 4.7
Consider the following velocity fields. Is it possible to represent these fields as the gradient of a scalar potential field? If so find the scalar potential field.
(a) $\vec{u} = -\omega y\ \hat{x} + \omega x\ \hat{y}$.
(b) $\vec{u} = \omega x\ \hat{x} - \omega y\ \hat{y}$.

(c) $\vec{u} = \alpha(y^2 - x^2)\ \hat{x} + 2\alpha xy\ \hat{y}$.
(d) $\vec{u} = \sinh(\kappa x)\cosh(\kappa y)\ \hat{x} - \cosh(\kappa x)\sinh(\kappa y)\ \hat{y}$.

§ 4.10 Boundary conditions

The top surface of the water is the interface between the air and the water. We represent this shape by the function $\eta(x, y, t)$ which is the vertical position of the surface of the water at the horizontal position (x, y) at time t. If the sea was completely flat then $\eta = 0$.

BC at a rigid surface

In a fluid in contact with a rigid boundary (such as the sea floor) the normal component of the velocity of the fluid must be zero

$$[\hat{n} \cdot \vec{u}]_{\text{surface}} = 0$$

where $\hat{n}$ is the normal to the surface.

BC at the top surface: kinematic

A particle at the surface stays at the surface. Let the trajectory of a particle at the surface be $\vec{r}(t)$. Then we can say that for all times t that

$$\hat{z} \cdot \vec{r} = \eta(\hat{x} \cdot \vec{r}, \hat{y} \cdot \vec{r}, t)$$

and thus also

$$\begin{aligned}
\frac{d}{dt}\hat{z} \cdot \vec{r} &= \frac{d}{dt}\eta(\hat{x} \cdot \vec{r}, \hat{y} \cdot \vec{r}, t) \\
\frac{d}{dt}\hat{z} \cdot \vec{r} &= \frac{\partial \eta}{\partial x}\frac{d}{dt}\hat{x} \cdot \vec{r} + \frac{\partial \eta}{\partial y}\frac{d}{dt}\hat{y} \cdot \vec{r} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial t}t \\
\hat{z} \cdot \frac{d\vec{r}}{dt} &= \frac{\partial \eta}{\partial x}\hat{x} \cdot \frac{d\vec{r}}{dt} + \frac{\partial \eta}{\partial y}\hat{y} \cdot \frac{d\vec{r}}{dt} + \frac{\partial \eta}{\partial t} \\
\hat{z} \cdot \vec{u} &= \frac{\partial \eta}{\partial x}\hat{x} \cdot \vec{u} + \frac{\partial \eta}{\partial y}\hat{y} \cdot \vec{u} + \frac{\partial \eta}{\partial t} \\
u_z &= \frac{\partial \eta}{\partial x}u_x + \frac{\partial \eta}{\partial y}u_y + \frac{\partial \eta}{\partial t} \\
\text{OR} \qquad \frac{\partial \phi}{\partial z} &= \frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial \eta}{\partial t}
\end{aligned}$$

This condition only applies at the top surface. To make this more clear we write.

$$\left[u_z - \frac{\partial \eta}{\partial x}u_x - \frac{\partial \eta}{\partial y}u_y\right]_{z=\eta} - \frac{\partial \eta}{\partial t} = 0$$

or

$$\left[\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial x} - \frac{\partial \eta}{\partial y}\frac{\partial \phi}{\partial y}\right]_{z=\eta} - \frac{\partial \eta}{\partial t} = 0$$

BC at the top surface: dynamic

The density of air is so much smaller than the density of water that differences in the pressure of the air from one place to another are usually insignificant. Thus in cases where the variation in air pressure are not significant, the pressure at the surface, of the water will be constant.

There is an additional force at the surface besides the pressure gradient and gravity. At the surface the cohesion of the water (the surface tension) tries to flatten the surface, like stretched rubber sheet wants to flatten out bumps. This causes the pressure on the water side of the interface not to be equal to the pressure on the air side. The pressure difference is $-\tau_s \nabla^2 \eta$, where τ_s is the surface tension of water. So that if we let the air pressure be zero then the pressure just inside the water is $P = -\tau_s \nabla^2 \eta$. With this we can use the dynamical equation

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla \phi|^2 + \frac{P}{\rho} + gz = \text{constant}$$

to find the condition dynamical boundary condition at the top surface.

$$\left[\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla \phi|^2\right]_{z=\eta} - \frac{\tau_s \nabla^2 \eta}{\rho} + g\eta = 0$$

The surface tension is expected to be insignificant except when the wavelength of the surface disturbance is much less than a meter. If the surface tension is negligible compared with the other terms then we have

$$\left[\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla \phi|^2\right]_{z=\eta} + g\eta = 0$$

Theorem: **Boundary Conditions**

If $\nabla \times \vec{u} = 0$ and $\nabla \cdot \vec{u} = 0$

$$[\hat{n} \cdot \nabla \phi]_{\text{surface}} = 0 \quad \text{Rigid}$$

$$\left[\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla \phi|^2\right]_{z=\eta} + g\eta = 0 \quad \text{Dynamic}$$

$$\left[\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial x} - \frac{\partial \eta}{\partial y}\frac{\partial \phi}{\partial y}\right]_{z=\eta} - \frac{\partial \eta}{\partial t} = 0 \quad \text{Kinematic}$$

The first of the boundary conditions is linear in the velocity potential field, but the last two are non-linear because they contain a product of two fields, for example $\frac{\partial \eta}{\partial y}\frac{\partial \phi}{\partial y}$. In the case that the displacement is small it is sometimes possible to ignore these product terms since if the field is small the square of the field will be smaller. In this case we get the following linearized boundary conditions.

Theorem: **Linearized Boundary Conditions**
If the movements are relatively small.

$$[\hat{n} \cdot \nabla \phi]_{\text{surface}} = 0 \quad \text{Rigid}$$

$$\left[\frac{\partial \phi}{\partial t}\right]_{z=0} + g\eta = 0 \quad \text{Dynamic}$$

$$\left[\frac{\partial \phi}{\partial z}\right]_{z=0} - \frac{\partial \eta}{\partial t} = 0 \quad \text{Kinematic}$$

§ 4.11 2-D Standing Wave

▷ Problem 4.8

Suppose that the water is trapped between vertical walls at $x = 0$ and $x = L$. Try a solution of the form

$$\phi = A\cos(\omega t)\cosh(k(z + d))\cos(kx - D)$$

where the bottom of the container is at to location $z = -d$.

(a) Show that the boundary condition at the bottom is satisfied.

(b) Applying the boundary conditions at the two rigid walls. What restrictions do you find on D and k?

(c) Apply the linearized boundary conditions at the top surface to find $\eta(x, t)$.

(d) Is there any restriction on the choice of ω?

§ 4.12 2-D Separable solutions to Laplace's equation

A common way to try and solve a partial differential equation is to try a function that is a product of single variable functions and see if the differential equation can be separated. This usually works pretty well for Laplace's equation.

Laplace's equation is a differential equation in the spacial variables (x, y, z). We will first look for a 2-D solution, one that has no dependence on y. To this end we propose a solution of the form $\phi(x, y, z, t) = f(x, t)h(z, t)$. Notice that there is no dependence on y so that $u_x = \frac{\partial \phi}{\partial y} = 0$, so this would be a motion in the $x - z$ plane. Now we write out Laplace's equation.

$$\nabla^2 \phi = \frac{\partial^2 f}{\partial x^2} h + f\frac{\partial^2 h}{\partial z^2} = 0$$

$$\longrightarrow \quad \frac{\frac{\partial^2 f}{\partial x^2}}{f} + \frac{\frac{\partial^2 h}{\partial z^2}}{h} = 0$$

$$\longrightarrow \quad \frac{\frac{\partial^2 h}{\partial z^2}}{h} = -\frac{\frac{\partial^2 f}{\partial x^2}}{f} = \text{constant}$$

Let us call the constant k^2 then we have to two equations

$$\frac{\partial^2 h}{\partial z^2} = k^2 h \qquad \text{AND} \qquad \frac{\partial^2 f}{\partial x^2} = -k^2 f$$

The solution to the first is

$$h = Ae^{kz} + Be^{-kz}$$

and the solution to the second is

$$f = C\cos(kx - D)$$

where A, B, C and D are possibly functions of time.

We pick our coordinate system so that $z = 0$ correspond to the mean sea level. If the water has a depth d then the sea floor is at the location $z = -d$. Assuming that the sea floor is flat then the normal the the bottom surface is $\hat{n} = \hat{z}$ and so

$$\begin{aligned}
[\hat{z}\cdot\nabla\phi]_{z=-d} = \left[\frac{\partial \phi}{\partial z}\right]_{z=-d} &= 0 \\
\longrightarrow \quad \left[\frac{\partial h}{\partial z}\right]_{z=-d} &= 0 \\
\longrightarrow \quad \left[kAe^{kz} - kBe^{-kz}\right]_{z=-d} &= 0 \\
\longrightarrow \quad Ae^{-kd} &= Be^{kd}
\end{aligned}$$

Using this we can rewrite the expression of h to remove B in favor of A.

$$\begin{aligned}
h &= Ae^{kz} + Be^{-kz} \\
&= Ae^{-kd}e^{k(z+d)} + Be^{kd}e^{-k(z+d)} \\
&= Ae^{-kd}\left(e^{k(z+d)} + e^{-k(z+d)}\right) \\
&= 2Ae^{-kd}\cosh(kz + kd)
\end{aligned}$$

Together then we have

$$\phi = 2Ae^{-kd}\cosh(kz + kd)C\cos(kx - D)$$

we might as well combine the constant $2Ae^{-kd}$ with the arbitrary function of time C to give $E = 2Ae^{-kd}C$ and

$$\phi = E\cosh(kz + kd)\cos(kx - D)$$

We will at times use the notation $\zeta = kz + kd$ and $\xi = kx - D$ then

$$\phi = E\cosh\zeta\cos\xi$$

Let us review what we have. For arbitrary functions $D(t)$ and $E(t)$ and constant k the function $\phi = E\cosh(k(z+d))\cos(kx-D)$ satisfies the boundary conditions at the sea floor and $\nabla^2\phi = 0$ throughout the body of the sea.

In the homework problems you will show that a sum of these functions with different values of k, D, and E,

$$\phi = \sum_{k,D,E} \phi_{k,D,E}$$

will also satisfy the boundary condition $\left[\hat{z}\cdot\nabla\phi\right]_{z=-d} = 0$ and $\nabla^2\phi = 0$, since both the boundary condition and the $\nabla^2\phi = 0$ are linear in ϕ.

▷ Problem 4.9
Let $\phi_1 = E_1\cosh(k_1(z+d))\cos(kx-D_1)$ and $\phi_2 = E_2\cosh(k_2(z+d))\cos(kx-D_2)$. Now let $\phi = a_1\phi_1 + a_2\phi_2$, with a_1 and a_2 arbitrary constants. Show that $\nabla^2\phi = 0$ and $\left[\hat{z}\cdot\nabla\phi\right]_{z=-d} = 0$.

▷ Problem 4.10
Show that if $\phi = E\cosh\zeta\cos\xi$ where $\zeta = k(z+d)$ and $\xi = kx - D$ then
(a) $\vec{u} = \nabla\phi = -kE\cosh\zeta\sin\xi\hat{x} + kE\sinh\zeta\cos\xi\hat{z}$.
(b) $|\nabla\phi|^2 = \nabla\phi\cdot\nabla\phi = k^2E^2\left(\sinh^2\zeta + \sin^2\xi\right)$
(c) $\frac{\partial\phi}{\partial t} = \cosh\zeta\left(\frac{dE}{dt}\cos\xi + E\frac{dD}{dt}\sin\xi\right)$

§ 4.13 2-D Traveling Wave for Linearized BC's

If we set $D = kvt$ and let E be a constant then we get a wave moving in the x direction without changing shape with a speed v.

$$\phi = E\cosh(k(z+d))\cos(k(x-vt))$$

For notational convenience we will use the notation $\omega = kv$. Then we can write

$$\phi = E\cosh(k(z+d))\cos(kx-\omega t) = E\cosh\zeta\cos\xi$$

with $\xi = kx - \omega t$.

▷ Problem 4.11
Use the linearized BC's to determine the surface shape η and the relationship between ω and k for the 2-D traveling wave.

▷ Problem 4.12
In the previous problem you found a relationship between the wave number k and the frequency ω for the 2-D traveling wave in the approximation that the linearized BC's are ok.
(a) Compute the phase velocity $v_p = \frac{\omega}{k}$ and .
(b) Compute the group velocity $\frac{\partial\omega}{\partial k}$.
(c) Show that if the water is very deep that $v_p = \sqrt{\frac{g}{k}}$.

(d) Show that $\frac{v_g}{v_p} = \frac{1}{2} + \frac{kd}{\sinh(2kd)}$.
(e) Define $v_\infty = \sqrt{\frac{g}{k}}$. Graph the ratio $\frac{v_p}{v_\infty}$ versus $\frac{d}{\lambda} = \frac{kd}{2\pi}$ for values from 0 to 1, (λ is the wavelength of the wave).
(f) Graph the ratio $\frac{v_g}{v_p}$ versus $\frac{d}{\lambda}$.

▷ PROBLEM 4.13
Do a contour plot of the velocity potential at $t = 0$ for $z = 0$ to $-d$ and for $x = 0$ to λ for the following cases. Add the vector field if you can figure out how to do so. For those using *Matlab* the function `quiver` will be very helpful for plotting the vector field.
(a) With $d = \lambda$.
(b) With $d = \lambda/2$.
(c) With $d = \lambda/5$.
(d) With $d = \lambda/10$.

▷ PROBLEM 4.14
Consider the case of shallow water.
(a) Show that in the limit that $kd \to 0$ that the phase velocity depends on the depth of the water but not the wavelength of the wave.
(b) Show that in the limit that $kd \to 0$ that the phase velocity and group velocity are the same.
(c) Graph the period versus the wavelength for wavelengths from 1 meter to 1000 metes and for water depths of (10, 30, 100, 300, ∞) meters.

Getting to know the motion

Let us understand the significance of the constant E. In the homework we found that with $\phi = E\cosh\zeta\cos\xi$ the surface is given by

$$\eta = -E\frac{\omega}{g}\cosh(kd)\sin\xi = -\frac{E}{\omega/k}\sinh(kd)\sin\xi = -\frac{E}{v_p}\sinh(kd)\sin\xi$$

So the surface traveling wave has an amplitude $a = \frac{E}{v_p}\sinh kd$. Thus we find that $E = \frac{a\ v_p}{\sinh(kd)}$ and we can then rewrite the velocity potential in terms of a.

$$\phi = a\ v_p\frac{\cosh\zeta}{\sinh(kd)}\cos\xi = \frac{a\omega}{k}\frac{\cosh\zeta}{\sinh(kd)}\cos\xi$$

and the velocity as

$$\begin{aligned}\vec{u} = \nabla\phi &= ka\ v_p\frac{-\cosh\zeta\sin\xi\hat{x} + \sinh\zeta\cos\xi\hat{z}}{\sinh(kd)}\\ &= a\omega\frac{-\cosh\zeta\sin\xi\hat{x} + \sinh\zeta\cos\xi\hat{z}}{\sinh(kd)}\end{aligned}$$

In the case that the water is deep this reduces to

$$\vec{u} = ka\ v_p e^{kz}\left[-\sin\xi\hat{x} + \cos\xi\hat{z}\right] = a\omega e^{kz}\left[-\sin\xi\hat{x} + \cos\xi\hat{z}\right]$$

Now let us consider the motion $\vec{r}(t) = r_x(t)\hat{x} + r_z(t)\hat{z}$ of a bug in the water. We know that

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{u}(r_x(\tau), r_z(\tau), \tau)d\tau$$

which is problematic to solve. We have already assumed that the amplitude of the motion a is small, so this means that $\vec{r}(t)$ does not move too much from $\vec{r}(0)$. For this reason we hope that $\vec{u}$ does not vary too much over the range the bugs movement, and we may approximate

$$\vec{u}(r_x(\tau), r_z(\tau), \tau) \approx \vec{u}(r_x(0), r_z(0), \tau) = \vec{u}(\tau)$$

Since all horizontal positions are equivalent we can choose WOLOG that $r_x(0) = 0$, then we can approximate

$$\vec{u}(t) \approx \frac{a\omega}{\sinh(kd)}\left[-\cosh\zeta_0 \sin\xi\hat{x} + \sinh\zeta_0\cos\xi\hat{z}\right]$$

with $\zeta_0 = kr_z(0) + kd$ and $\xi = -\omega t$. and

$$\begin{aligned}
\vec{r}(t) &\approx \vec{r}(0) + \int_0^t \vec{u}(r_x(0), r_z(0), \tau)d\tau \\
&= \vec{r}(0) + \int_0^t \frac{a\omega}{\sinh(kd)}\left[-\cosh\zeta_0\sin\xi_\tau\ \hat{x} + \sinh\zeta_0\cos\xi_\tau\ \hat{z}\right]d\tau \\
&= \vec{r}(0) - \frac{a}{\sinh(kd)}\left[\cosh\zeta_0\cos\xi_\tau\ \hat{x} + \sinh\zeta_0\sin\xi_\tau\ \hat{z}\right]_0^t \\
&= \vec{r}(0) + \frac{a\cosh\zeta_0}{\sinh(kd)}\hat{x} - a\frac{\cosh\zeta_0\cos\xi\ \hat{x} + \sinh\zeta_0\sin\xi\ \hat{z}}{\sinh(kd)}
\end{aligned}$$

The first part $\vec{r}(0) + \frac{a\cosh\zeta_0}{\sinh(kd)}\hat{x}$ is a constant so let us call it $\vec{C}$. Then we can write.

$$\vec{r}(t) \approx \vec{C} - a\frac{\cosh\zeta_0\cos\xi\ \hat{x} + \sinh\zeta_0\sin\xi\ \hat{z}}{\sinh(kd)}$$

If we let $R_x = a\frac{\cosh\zeta_0}{\sinh(kd)}$ and $R_z = a\frac{\sinh\zeta_0}{\sinh(kd)}$ we can write

$$\vec{r}(t) \approx \vec{C} - R_x\cos\xi\ \hat{x} - R_z\sin\xi\ \hat{z}$$

which is an elipse about the center $\vec{C}$.

▷ PROBLEM 4.15

Integrate the velocity numerically in order to find the trajectory. To make things easier do the deep water approximation for the velocity field. The simulation should show what is called the *Stokes drift* velocity, that is the fact that the water has a net motion in the direction of the wave travel. There are extensive suggestions for this problem in the hints section. When the code is working run it for a range of values of ka and compute the ratio of the drift velocity and phase velocity and make a graph of this velocity ratio versus ka.

▷ PROBLEM 4.16
Consider the deep water limit $d \gg \lambda$ of the velocity field.
(a) What is the velocity of the water in the trough of a wave?
(b) What is the velocity of the water at the peak of the wave?
(c) On what does the ratio of the velocity and phase velocity depend?

Theorem: **Relationships for linear waves:** $k\eta \ll 1$
Starting from the dispersion relation $\omega^2 = gk\tanh(kd)$ we find the following.

$$v_p = \sqrt{\frac{g\tanh(kd)}{k}}$$

$$v_\infty = \lim_{kd\to\infty} v_p = \sqrt{\frac{g}{k}} \qquad \text{AND} \qquad \frac{v_p}{v_\infty} = \sqrt{\tanh(kd)}$$

$$v_0 = \lim_{kd\to 0} v_p = \sqrt{gd} \qquad \text{AND} \qquad \frac{v_p}{v_0} = \sqrt{\frac{\tanh(kd)}{kd}}$$

$$\frac{v_g}{v_p} = \frac{1}{2} + \frac{kd}{\sinh(2kd)}$$

§ 4.14 Sanity check

We have been using the linearized boundary conditions. We have not done a good job of justifying why this would be ok, nor do we have a concrete way of deciding when it will not be ok. We should do that now.

First let us review what we found in our linear approximation.

$$\begin{aligned}
\omega^2 &= gk\tanh(kd)\\
\eta &= -a\sin\xi\\
\phi &= \frac{a\omega}{k}\frac{\cosh\zeta}{\sinh(kd)}\cos\xi\\
\nabla\phi &= a\omega\frac{-\cosh\zeta\sin\xi\hat{x} + \sinh\zeta\cos\xi\hat{z}}{\sinh(kd)}\\
|\nabla\phi|^2 &= a^2\omega^2\frac{\sinh^2\zeta + \sin^2\xi}{\sinh^2(kd)}
\end{aligned}$$

Now we want to see how far off the two BC's are.

$$\left[\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2\right]_{z=\eta} = -g\eta \quad \text{Dynamic}$$

$$\left[\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x}\right]_{z=\eta} = \frac{\partial\eta}{\partial t} \quad \text{Kinematic}$$

To make things a little easier to deal with we will instead work with the two equations

$$\frac{1}{ag}\left[\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2\right]_{z=\eta} = -\frac{\eta}{a} \quad \text{Dynamic}$$

$$\frac{1}{a\omega}\left[\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x}\right]_{z=\eta} = \frac{1}{a\omega}\frac{\partial\eta}{\partial t} \quad \text{Kinematic}$$

▷ PROBLEM 4.17
Compute the left hand side and the right hand side of the dynamic boundary condition. Plot both sides as a function of ξ. Let $d = \lambda/2$. Are the LHS and RHS the same? Repeat for $ka = 0.001, 0.01$, and 0.1. Now do the same for the kinematic boundary condition.

§ 4.15 Pressure on the bottom

Using the 2D traveling wave velocity potential we evaluate the pressure at the bottom surface, $z = -d$. We have

$$\phi = \frac{a\omega}{k}\frac{\cosh\zeta}{\sinh(kd)}\cos\xi$$

So we can use the dynamical equation $\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{P}{\rho} + gz = 0$ to find the pressure. We notice first that $\zeta = kz + kd$ so at $z = -d$ we see that $\zeta = 0$.

$$\begin{aligned}
\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + \frac{P}{\rho} + gz &= 0 \\
\longrightarrow \quad \frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + gz &= -\frac{P}{\rho} \\
\frac{a\omega^2}{k}\frac{\cosh\zeta}{\sinh(kd)}\sin\xi + \tfrac{1}{2}a^2\omega^2\frac{\sinh^2\zeta + \sin^2\xi}{\sinh^2(kd)} + gz &= -\frac{P}{\rho} \\
ag\tanh(kd)\frac{\cosh\zeta}{\sinh(kd)}\sin\xi + \tfrac{1}{2}ka^2 g\tanh(kd)\frac{\sinh^2\zeta + \sin^2\xi}{\sinh^2(kd)} + gz &= -\frac{P}{\rho} \\
a\frac{\cosh\zeta}{\cosh(kd)}\sin\xi + \tfrac{1}{2}ka^2\frac{\sinh^2\zeta + \sin^2\xi}{\cosh(kd)\sinh(kd)} + z &= -\frac{P}{\rho g}
\end{aligned}$$

Now evaluating at $z = -d$ we find

$$\begin{aligned}
\frac{P}{\rho g} &= d - a\frac{1}{\cosh(kd)}\sin\xi - \tfrac{1}{2}ka^2\frac{0+\sin^2\xi}{\cosh(kd)\sinh(kd)} \\
&= d - \frac{a\sin\xi}{\cosh(kd)}\left[1 + \tfrac{1}{2}\frac{ka\sin\xi}{\sinh(kd)}\right] \\
&= d - \frac{-\eta}{\cosh(kd)}\left[1 + \tfrac{1}{2}\frac{-k\eta}{\sinh(kd)}\right] \\
&= d + \frac{\eta}{\cosh(kd)}\left[1 - \tfrac{1}{2}\frac{k\eta}{\sinh(kd)}\right] \\
&\approx d + \frac{\eta}{\cosh(kd)}
\end{aligned}$$

5 Beyond linearization

§ 5.1 Deep water expansion

Here we note that even at a modest depth of $d = \frac{\lambda}{2}$ that $e^{-2kd} = e^{-2\pi} = 0.001867$ is a small number. Thus we are lead to the usefulness of the following expansion for the case that the depth is greater than half the wavelength.

$$\begin{aligned}
\frac{\cosh(kz + kd)}{\sinh(kd)} &= \frac{e^{kz+kd} + e^{-kz-kd}}{e^{kd} - e^{-kd}} \\
&= \frac{e^{kz} + e^{-kz}e^{-2kd}}{1 - e^{-2kd}} \\
&= (e^{kz} + e^{-kz}e^{-2kd})(1 + e^{-2kd} + e^{-4kd} + \cdots)
\end{aligned}$$

So if we can ignore things of size e^{-2kd} compared with 1 then we can approximate $\frac{\cosh(kz+kd)}{\sinh(kd)} \approx e^{kz}$. You can do similarly for

Theorem: **Cosh and Sinh deep water expansion**

$$\frac{\cosh(kz + kd)}{\sinh(kd)} = (e^{kz} + e^{-kz}e^{-2kd})(1 + e^{-2kd} + e^{-4kd} + \cdots)$$

and

$$\frac{\sinh(kz + kd)}{\sinh(kd)} = (e^{kz} - e^{-kz}e^{-2kd})(1 + e^{-2kd} + e^{-4kd} + \cdots)$$

So if we have $d > \lambda/2$ we can approximate.

$$\phi = a\frac{\omega}{k}e^{kz}\cos\xi$$

§ 5.2 Traveling wave assumption: recasting BC's in ξ

If we assume that the surface wave $\eta(x, t)$ moves without changing shape and that it moves with the same velocity as the disturbance of the velocity potential then η is only a function of the combination $\xi = kx - \omega t$. Because of this

$$\frac{\partial \eta}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial \eta}{\partial \xi} = -\omega\frac{\partial \eta}{\partial \xi}$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial \xi} = k\frac{\partial \eta}{\partial \xi}$$

$$\frac{\partial\eta}{\partial y} = \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial\xi} = 0\frac{\partial\eta}{\partial\xi}$$

Substituting this into the kinematic BC $\left[\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x}\right]_{z=\eta} = \frac{\partial\eta}{\partial t}$ gives us

$$\left[\frac{\partial\phi}{\partial z} - k\frac{\partial\eta}{\partial\xi}\frac{\partial\phi}{\partial x}\right]_{z=\eta} = -\omega\frac{\partial\eta}{\partial\xi}$$

$$\left[k\frac{\partial\phi}{\partial\zeta} - k^2\frac{\partial\eta}{\partial\xi}\frac{\partial\phi}{\partial\xi}\right]_{z=\eta} = -\omega\frac{\partial\eta}{\partial\xi}$$

Theorem: **Boundary Conditions: recast**

$$\left[-\omega\frac{\partial\phi}{\partial\xi} + \tfrac{1}{2}|\nabla\phi|^2\right]_{z=\eta} + g\eta = 0 \quad \text{Dynamic}$$

$$\left[k\frac{\partial\phi}{\partial\zeta} - k^2\frac{\partial\eta}{\partial\xi}\frac{\partial\phi}{\partial\xi}\right]_{z=\eta} + \omega\frac{\partial\eta}{\partial\xi} = 0 \quad \text{Kinematic}$$

It is important to note that once we have substituted $z = \eta$ that the boundary conditions only have the variable ξ remaining. This is because where we had z before we now have η and η is a function of ξ.

So for example substituting $\phi = a\frac{\omega}{k}\frac{\cosh\zeta}{\sinh(kd)}\cos\xi$ into the kinematic equation we find

$$\frac{\sinh\zeta_\eta}{\sinh(kd)}\cos\xi + ka\frac{\cosh\zeta_\eta}{\sinh(kd)}\sin\xi\frac{1}{a}\frac{\partial\eta}{\partial\xi} = -\frac{1}{a}\frac{\partial\eta}{\partial\xi}$$

We see that this is only a function ξ.

Even so it looks a *little* difficult to solve, the η is buried inside of the ζ_η inside a $\coth\zeta_\eta$ and a $\sinh\zeta_\eta$. :/

It must be **Approximation Time**!

§ 5.3 Small amplitude limit of 1-D Traveling Wave: $k\eta \ll 1$

Let us look at $\cosh\zeta_\eta$. If η is small compared with the wavelength $\lambda = \frac{2\pi}{k}$ we can make the following expansion.

$$\cosh\zeta_\eta = \cosh(k(\eta + d)) = \cosh(kd) + \sinh(kd)k\eta + \mathcal{O}\left[k^2\eta^2\right]$$

Similarly

$$\sinh\zeta_\eta = \sinh(k(\eta + d)) = \sinh(kd) + \cosh(kd)k\eta + \mathcal{O}\left[k^2\eta^2\right]$$

So if $k\eta \ll 1$ we can approximate

$$\frac{\cosh\zeta_\eta}{\sinh(kd)} \approx \coth(kd) + k\eta$$

$$\frac{\sinh \zeta_\eta}{\sinh(kd)} \approx 1 + \tanh(kd)k\eta$$

so we can approximate for $k\eta \ll 1$ that

$$(1 + ka(\coth(kd) + k\eta)\sin\xi)\,\frac{1}{a}\frac{\partial \eta}{\partial \xi} + (1 + \tanh(kd)k\eta)\cos\xi = 0$$

Now recall that a is the amplitude of the motion near the surface so a and η are of the same order. Thus the term $kak\eta\sin\xi$ is of order $(k\eta)^2$ and it is added to 1 so it must be ignored also since already ignored order $(k\eta)^2$ compared with 1. This leaves us with

$$(1 + ka\coth(kd)\sin\xi)\,\frac{1}{a}\frac{\partial \eta}{\partial \xi} + (1 + \tanh(kd)k\eta)\cos\xi = 0$$

or

$$\frac{1}{a}\frac{\partial \eta}{\partial \xi} = -\frac{(1 + \tanh(kd)k\eta)\cos\xi}{1 + ka\coth(kd)\sin\xi}$$

If we also ignore order $k\eta$ in comparison with 1 then we have

$$\frac{1}{a}\frac{\partial \eta}{\partial \xi} = -\cos\xi \;\longrightarrow\; \frac{\eta}{a} = -\sin\xi$$

Thus we have the zero order approximation.

$$\frac{\eta}{a} = -\sin\xi$$
$$\phi = a\frac{\omega}{k}\frac{\cosh\zeta}{\sinh(kd)}\cos\xi$$

But we have not verified that the dynamic BC is ok.

$$\left[\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2\right]_{z=\eta} = -g\eta$$

is satisfied. Let us try substituting and see what happens.

$$\left[-\omega\frac{\partial \phi}{\partial \xi} + \tfrac{1}{2}\left(k\frac{\partial \phi}{\partial \xi}\right)^2 + \tfrac{1}{2}\left(k\frac{\partial \phi}{\partial \zeta}\right)^2\right]_{z=\eta} = -g\eta$$

$$a\frac{\omega^2}{k}\frac{\cosh\zeta_\eta}{\sinh(kd)}\sin\xi + a^2\omega^2\frac{\cosh(2\zeta_\eta) - \cos(2\xi)}{4\sinh^2(kd)} = -g\eta$$

$$\frac{\cosh\zeta_\eta}{\sinh(kd)}\sin\xi + ka\frac{\cosh(2\zeta_\eta) - \cos(2\xi)}{4\sinh^2(kd)} = -\frac{gk}{\omega^2}\frac{\eta}{a}$$

$$(\coth(kd) + k\eta)\sin\xi + ka\frac{\cosh(2\zeta_\eta) - \cos(2\xi)}{4\sinh^2(kd)} = -\frac{gk}{\omega^2}\frac{\eta}{a}$$

$$\coth(kd)\sin\xi = -\frac{gk}{\omega^2}\frac{\eta}{a}$$

$$-\sin\xi = \frac{gk\tanh(kd)}{\omega^2}\frac{\eta}{a}$$

So we see that this BC give the same results $\frac{\eta}{a} = -\sin\xi$ if again we ignore $k\eta$ in comparison with 1 and if $\frac{gk\tanh(kd)}{\omega^2} = 1$, that is if

$$\omega^2 = gk\tanh(kd)$$

.

§ 5.4 Deep water limit of 1-D traveling wave: $d > \lambda/2$

No we take a different limiting approximation. In this case we do not make assumptions about the amplitude of the waves but about the depth of the water. Starting again with an exactly solution to to $\nabla^2\phi = 0$

$$\phi = a\frac{\omega}{k}\frac{\cosh\zeta}{\sinh(kd)}\cos\xi$$

we take the case that $d > \lambda/2 \longrightarrow kd > \pi$. Consider

$$\frac{\cosh\zeta_\eta}{\sinh(kd)} = \frac{\cosh(kd + k\eta)}{\sinh(kd)} = \frac{e^{kd}e^{k\eta} + e^{-kd}e^{-k\eta}}{e^{kd} - e^{-kd}} = \frac{e^{k\eta} + e^{-2kd}e^{-k\eta}}{1 - e^{-2kd}}$$

Now if $kd > \pi$ is large then $\epsilon = e^{-2kd} < 0.002$. For this reason we are lead to approximate

$$\frac{\cosh\zeta_\eta}{\sinh(kd)} = e^{k\eta}\left[1 + e^{-2kd}\frac{1 + e^{-2k\eta}}{1 - e^{-2kd}}\right] \approx e^{k\eta}$$

Even the largest possible waves have $|k\eta| < 0.6$. Thus with $kd > \pi$ we have that the error term is

$$e^{-2kd}\frac{1 + e^{-2k\eta}}{1 - e^{-2kd}} < 0.0035$$

so we make an error of at most 0.3% and we can safely approximate $\frac{\cosh\zeta_\eta}{\sinh(kd)} \approx e^{k\eta}$ Thus in the deep water limit we have

$$\begin{aligned}
\phi &= \frac{a\omega}{k}e^{kz}\cos\xi \\
\left[\frac{\partial\phi}{\partial z}\right]_{z=\eta} &= a\omega e^{k\eta}\cos\xi \\
\left[\frac{\partial\phi}{\partial x}\right]_{z=\eta} &= -a\omega e^{k\eta}\sin\xi \\
\left[\frac{\partial\phi}{\partial t}\right]_{z=\eta} &= \frac{a\omega^2}{k}e^{k\eta}\sin\xi
\end{aligned}$$

Now let us consider the dynamic boundary condition again.

$$\left[\frac{\partial\phi}{\partial t} + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2\right]_{z=\eta} = -g\eta$$

$$a\frac{\omega^2}{k}e^{k\eta}\sin\xi + \tfrac{1}{2}\left(a\omega e^{k\eta}\sin\xi\right)^2 + \tfrac{1}{2}\left(a\omega e^{k\eta}\cos\xi\right)^2 = -g\eta$$

$$a\frac{\omega^2}{k}e^{k\eta}\sin\xi + \tfrac{1}{2}a^2\omega^2 e^{2k\eta} = -g\eta$$

$$e^{k\eta}\sin\xi + \tfrac{1}{2}ka\ e^{2k\eta} = -\frac{gk}{\omega^2}\frac{\eta}{a}$$

$$\frac{\omega^2}{gk}e^{k\eta}\left(\sin\xi + \tfrac{1}{2}ka\ e^{k\eta}\right) + \frac{\eta}{a} = 0$$

Now the kinematic BC

$$\left[\frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x}\right]_{z=\eta} = \frac{\partial\eta}{\partial t}$$

$$ka\frac{\omega}{k}e^{k\eta}\cos\xi + k^2\frac{\partial\eta}{\partial\xi}a\frac{\omega}{k}e^{k\eta}\sin\xi = -\omega\frac{\partial\eta}{\partial\xi}$$

$$e^{k\eta}\cos\xi + k\ e^{k\eta}\sin\xi\frac{\partial\eta}{\partial\xi} + \frac{1}{a}\frac{\partial\eta}{\partial\xi} = 0$$

$$\frac{\partial}{\partial\xi}\left[e^{k\eta}\sin\xi + \frac{\eta}{a}\right] = 0$$

$$e^{k\eta}\sin\xi + \frac{\eta}{a} = K$$

where K is a constant.

Theorem: **Deep water BC's for 1-D traveling wave**
In the case that $d > \lambda/2$ we can approximate the top surface boundary conditions for the form $\phi = a\frac{\omega}{k}\frac{\cosh\zeta}{\sinh(kd)}\cos\xi$ as

$$\frac{\omega^2}{gk}e^{k\eta}\left(\sin\xi + \tfrac{1}{2}ka\ e^{k\eta}\right) + \frac{\eta}{a} = 0 \quad \text{dynamic}$$

$$e^{k\eta}\sin\xi + \frac{\eta}{a} = K \quad \text{kinematic}$$

Deep water limit: amplitude expansion to order $(ka)^0$

If we take the boundary conditions and ignore terms of order ka compared with terms of order 1 then we have

$$\frac{\omega^2}{gk}e^{k\eta}\sin\xi + \frac{\eta}{a} = 0 \quad \text{dynamic}$$

$$e^{k\eta}\sin\xi + \frac{\eta}{a} = K \quad \text{kinematic}$$

But we also know that if ka is small we can approximate $e^{k\eta} = e^{ka\frac{\eta}{a}} = 1 + ka\frac{\eta}{a} + \mathcal{O}\left[(ka)^2\right] = 1 + \mathcal{O}\left[(ka)^1\right]$. So since we are dropping terms of order ka when added to terms or order 1 we can approximate the BCs as

$$\begin{aligned} \frac{\omega^2}{gk}\sin\xi + \frac{\eta}{a} &= 0 \quad \text{dynamic} \\ \sin\xi + \frac{\eta}{a} &= K \quad \text{kinematic} \end{aligned}$$

These are the same thing if we pick $K = 0$ and $\omega^2 = gk$ in which case we get $\eta = -a\sin\xi$. Which is the solution we go by using the linearized BCs. So we see that the linearized BC's have tossed terms of order ka, and only gives the strictly correct solution in the case that the amplitude is zero or very close to it.

Deep water limit to order $(ka)^1$

Now we do the same thing but keep order ka and toss order $(ka)^2$. Let us see how this works out.

First we note that $e^{k\eta} \approx 1 + ka\frac{\eta}{a}$, so that the above two equations can be written to first order as follows.

$$\begin{aligned} \frac{\omega^2}{gk}(1 + ka\frac{\eta}{a})\left(\sin\xi + \tfrac{1}{2}ka(1 + ka\frac{\eta}{a})\right) + \frac{\eta}{a} &= 0 \quad \text{dynamic} \\ (1 + ka\frac{\eta}{a})\sin\xi + \frac{\eta}{a} &= K \quad \text{kinematic} \end{aligned}$$

Now tossing further $(ka)^2$ terms we have

$$\begin{aligned} \frac{\omega^2}{gk}\left(\sin\xi + ka\frac{\eta}{a}\sin\xi + \tfrac{1}{2}ka\right) + \frac{\eta}{a} &= 0 \quad \text{dynamic} \\ (1 + ka\frac{\eta}{a})\sin\xi + \frac{\eta}{a} &= K \quad \text{kinematic} \end{aligned}$$

Rewriting slightly we have

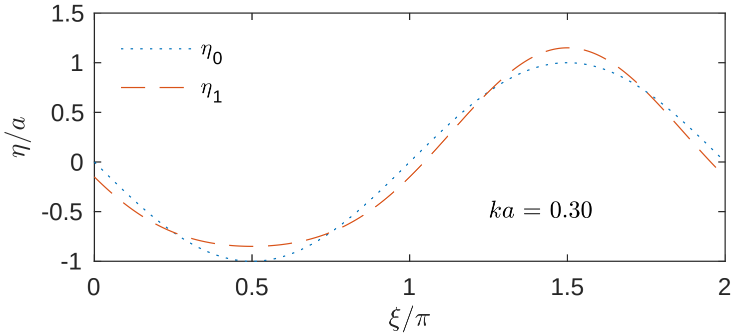
$$\begin{aligned} \frac{\omega^2}{gk}\left(1 + ka\frac{\eta}{a}\right)\sin\xi + \frac{\eta}{a} &= -\frac{\omega^2}{gk}\tfrac{1}{2}ka \quad \text{dynamic} \\ (1 + ka\frac{\eta}{a})\sin\xi + \frac{\eta}{a} &= K \quad \text{kinematic} \end{aligned}$$

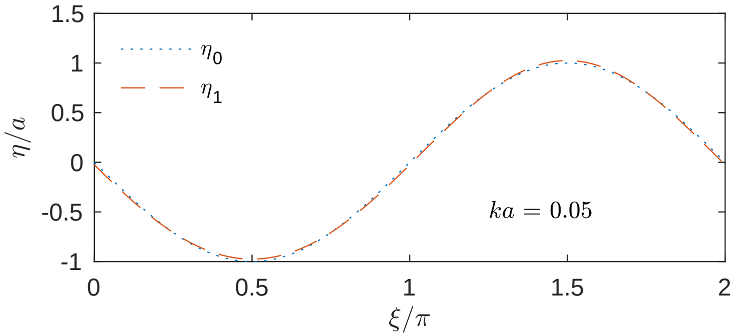
So we see that the BC's agree if we pick the constant to be $K = -\frac{1}{2}ka$ and if we pick the frequency $\omega^2 = gk$. Solving for $\frac{\eta}{a}$ we find

$$\frac{\eta}{a} = -\frac{\sin\xi + \frac{1}{2}ka}{1 + ka\sin\xi}$$

This can be written in a slightly different form as follows which is not equal but equal to order $(ka)^2$.

$$\begin{aligned}\frac{\eta}{a} &= -\frac{\sin\xi + \frac{1}{2}ka}{1 + ka\sin\xi} + \mathcal{O}\left[(ka)^2\right] \\ &= -\sin\xi - \tfrac{1}{2}ka(1 - 2\sin^2\xi) + \mathcal{O}\left[(ka)^2\right] \\ &= -\sin\xi - \tfrac{1}{2}ka\cos 2\xi + \mathcal{O}\left[(ka)^2\right]\end{aligned}$$





§ 5.5 Deep water limit with $(ka)^3 \ll 1$

$$(1 + k\eta + \tfrac{1}{2}k^2\eta^2)\sin\xi + \tfrac{1}{2}ka(1 + 2k\eta) + \frac{gk}{\omega^2}\frac{\eta}{a} = 0 \quad \text{dynamic}$$

$$(1 + k\eta + \tfrac{1}{2}k^2\eta^2)\sin\xi + \frac{\eta}{a} = K \quad \text{kinematic}$$

or

$$\tfrac{1}{2}k^2a^2\sin\xi\frac{\eta^2}{a^2} + \left(\frac{gk}{\omega^2} + ka\sin\xi + k^2a^2\right)\frac{\eta}{a} = -\sin\xi - \tfrac{1}{2}ka \quad \text{dynamic}$$

$$\tfrac{1}{2}k^2a^2\sin\xi\frac{\eta^2}{a^2} + (1 + ka\sin\xi)\frac{\eta}{a} = -\sin\xi + K \quad \text{kinematic}$$

Which are consistent if we let the constant $K = -\frac{1}{2}ka$ and the frequency be such that

$$\frac{gk}{\omega^2} + k^2a^2 = 1 \longrightarrow \omega^2 = \frac{gk}{1-k^2a^2}$$

then the two BC's are both

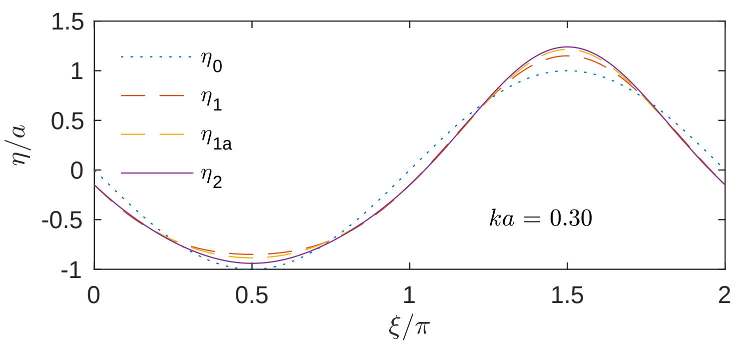
$$\tfrac{1}{2}k^2a^2 \sin\xi \frac{\eta^2}{a^2} + (1 + ka\sin\xi)\,\frac{\eta}{a} = -\sin\xi - \tfrac{1}{2}ka$$

Solving the quadratic gives

$$\frac{\eta}{a} = \frac{\sqrt{(1+ka\sin\xi)^2 - 2k^2a^2\sin^2\xi - k^3a^3\sin\xi} - (1+ka\sin\xi)}{k^2a^2\sin\xi}$$

which is a little messy looking, but if we recall that is is only good up to order $(ka)^2$ then we can do a power series expansion of the square root and then we find that

$$\begin{aligned}
\frac{\eta}{a} &= -\sin\xi - \tfrac{1}{2}ka(1-2\sin^2\xi) + \frac{k^2a^2}{2}(\sin\xi - 3\sin^3\xi) + \mathcal{O}\left[(ka)^3)\right] \\
&= -\sin\xi - \tfrac{1}{2}ka\cos 2\xi + \frac{k^2a^2}{8}(-5\sin\xi + 3\sin 3\xi) + \mathcal{O}\left[(ka)^3)\right] \\
&= -\left(1 + \frac{5}{8}(ka)^2\right)\sin\xi - \tfrac{1}{2}ka\cos 2\xi + \frac{5}{8}(ka)^2\sin 3\xi + \mathcal{O}\left[(ka)^3)\right]
\end{aligned}$$



▷ Problem 5.1

Check the accuracy of the the zero order, first order, and second order approximations for the deep water case by evaluating the error in the BCs. Take the BCs in the form

$$\frac{\omega^2}{gk}e^{k\eta}\left[\sin\xi + \tfrac{1}{2}ka\ e^{k\eta}\right] + \frac{\eta}{a} = 0 \quad \text{dynamic}$$

$$-K + e^{k\eta}\sin\xi + \frac{\eta}{a} = 0 \quad \text{kinematic}$$

and plot the LHS for all three orders on the same graph. Use $ka = 0.2$. How close are they to zero?

(a) The zero order solution: $K = 0$, $\frac{\omega^2}{gk} = 1$, and $\frac{\eta}{a} = -\sin\xi$.

The first order solution: $K = -\frac{1}{2}ka$, $\frac{\omega^2}{gk} = 1$, and

$$\frac{\eta}{a} = -\frac{\sin\xi + \frac{1}{2}ka}{1 + ka\sin\xi}.$$

The second order solution: $K = -\frac{1}{2}ka$, $\frac{\omega^2}{gk} = \frac{1}{1-k^2a^2}$, and

$$\frac{\eta}{a} = -\sin\xi - \tfrac{1}{2}ka(1 - 2\sin^2\xi) + \tfrac{1}{2}k^2a^2(\sin\xi - 3\sin^3\xi).$$

(b) In the second order approximation we had the alternate expression

$$\frac{\eta}{a} = -\frac{(1 + ka\sin\xi) - \sqrt{(1 + ka\sin\xi)^2 - 2k^2a^2\sin^2\xi - k^3a^3\sin\xi}}{k^2a^2\sin\xi}.$$

What could cause problems in evaluating numerically this expression?

§ 5.6 Try the same without deep water limit

We had the kinetic BC

$$\frac{\sinh\zeta_\eta}{\sinh(kd)}\cos\xi + ka\frac{\cosh\zeta_\eta}{\sinh(kd)}\sin\xi\frac{1}{a}\frac{\partial\eta}{\partial\xi} + \frac{1}{a}\frac{\partial\eta}{\partial\xi} = 0$$

We now notice that the first two terms together is a total derivative. Thus we can write

$$\frac{\partial}{\partial\xi}\left[\frac{\sinh\zeta_\eta}{\sinh(kd)}\sin\xi\right] + \frac{1}{a}\frac{\partial\eta}{\partial\xi} = 0$$

or

$$\frac{\partial}{\partial\xi}\left[\frac{\sinh\zeta_\eta}{\sinh(kd)}\sin\xi + \frac{\eta}{a}\right] = 0$$

and thus

$$\frac{\sinh\zeta_\eta}{\sinh(kd)}\sin\xi + \frac{\eta}{a} = K$$

where K is a constant.

While the dynamic BC is

$$\left[-\omega\frac{\partial\phi}{\partial\xi} + \frac{1}{2}|\nabla\phi|^2\right]_{z=\eta} + g\eta = 0$$

$$\frac{a\omega^2}{k}\frac{\cosh\zeta_\eta}{\sinh(kd)}\sin\xi + \frac{a^2\omega^2}{2}\frac{\sinh^2\zeta_\eta + \sin^2\xi}{\sinh^2(kd)} + g\eta = 0$$

$$\frac{\cosh\zeta_\eta}{\sinh(kd)}\sin\xi + \frac{ka}{2}\frac{\sinh^2\zeta_\eta + \sin^2\xi}{\sinh^2(kd)} + \frac{gk}{\omega^2}\frac{\eta}{a} = 0$$

So our BC's are

Theorem: **BC's for separable velocity potential**

$$\frac{\cosh \zeta_\eta}{\sinh(kd)} \sin\xi + \frac{ka}{2}\,\frac{\sinh^2\zeta_\eta + \sin^2\xi}{\sinh^2(kd)} + \frac{gk}{\omega^2}\frac{\eta}{a} = 0 \quad \text{dynamic}$$

$$\frac{\sinh \zeta_\eta}{\sinh(kd)} \sin\xi + \frac{\eta}{a} = K \quad \text{kinetic}$$

Now we expand the hyperbolic functions in powers of $k\eta$,

$$\begin{aligned}
\frac{\cosh\zeta_\eta}{\sinh kd} &= \frac{e^{k\eta} + \epsilon e^{-k\eta}}{1-\epsilon} \\
&= \frac{\sum_n \frac{(k\eta)^n}{n!} + \epsilon \sum_n \frac{(-k\eta)^n}{n!}}{1-\epsilon} \\
&= \frac{\sum_n^{\text{even}} \frac{(k\eta)^n}{n!} + \sum_n^{\text{odd}} \frac{(k\eta)^n}{n!} + \epsilon \sum_n^{\text{even}} \frac{(k\eta)^n}{n!} - \epsilon \sum_n^{\text{odd}} \frac{(k\eta)^n}{n!}}{1-\epsilon} \\
&= \frac{(1+\epsilon)\sum_n^{\text{even}} \frac{(k\eta)^n}{n!} + (1-\epsilon)\sum_n^{\text{odd}} \frac{(k\eta)^n}{n!}}{1-\epsilon} \\
&= \frac{1+\epsilon}{1-\epsilon} \sum_n^{\text{even}} \frac{(k\eta)^n}{n!} + \sum_n^{\text{odd}} \frac{(k\eta)^n}{n!} \\
&= \coth(kd) \sum_n^{\text{even}} \frac{(k\eta)^n}{n!} + \sum_n^{\text{odd}} \frac{(k\eta)^n}{n!} \\
\frac{\sinh\zeta_\eta}{\sinh kd} &= \frac{e^{k\eta} - \epsilon e^{-k\eta}}{1-\epsilon} \\
&= \frac{(1-\epsilon)\sum_n^{\text{even}} \frac{(k\eta)^n}{n!} + (1+\epsilon)\sum_n^{\text{odd}} \frac{(k\eta)^n}{n!}}{1-\epsilon} \\
&= \sum_n^{\text{even}} \frac{(k\eta)^n}{n!} + \coth(kd) \sum_n^{\text{odd}} \frac{(k\eta)^n}{n!}
\end{aligned}$$

Now let us use this expansion to get the second order in ka expansion of the BC's. We will use the notation that $C = \coth(kd)$, $T = \tanh(kd)$ and $\alpha = ka$ and $\mu = \frac{\eta}{a}$. We start with the dynamic BC.

$$\left[C + \alpha\mu + \frac{C}{2}\alpha^2\mu^2\right]\sin\xi + \frac{\alpha}{2}\left[1 + 2C\alpha\mu + \frac{\sin^2\xi}{\sinh^2(kd)}\right] + \frac{gk}{\omega^2}\mu = 0$$

$$\left(C\sin\xi + \frac{\alpha}{2} + \frac{\alpha}{2}\frac{\sin^2\xi}{\sinh^2(kd)}\right) + \left(\frac{gk}{\omega^2} + \alpha\sin\xi + C\alpha^2\right)\mu + \frac{C}{2}\alpha^2\sin\xi\mu^2 = 0$$

$$\left(\sin\xi + \frac{\alpha}{2}T + \alpha\frac{\sin^2\xi}{\sinh(2kd)}\right) + \left(\frac{gk}{\omega^2}T + \alpha^2 + \alpha T\sin\xi\right)\mu + \frac{\alpha^2}{2}\sin\xi\mu^2 = 0$$

Now the kinetic BC $\frac{\sinh\zeta_\eta}{\sinh(kd)}\sin\xi + \frac{\eta}{a} = K$ becomes

$$\left(1 + C\alpha\mu + \frac{1}{2}\alpha^2\mu^2\right)\sin\xi + \mu = K$$

or

$$(\sin\xi - K) + (1 + \alpha C\sin\xi)\,\mu + \frac{\alpha^2}{2}\sin\xi\ \mu^2 = 0$$

These two are only equivalent if we ignore both the α and the α^2 terms, in which case we have that they are equivalent if $\omega^2 = gk\tanh(kd)$ and $K = 0$.

They are exactly equal if we let $kd \to \infty$ in which case $C = T = 1$, but this is our second order deep water solution.

It is tempting to get the first order finite depth effect, so let us see if this goes anywhere. Let $\epsilon = e^{-2kd}$. Note that if $kd > 2.3$ then $\epsilon < 0.01$, so it doesn't take much depth to make ϵ small. To first order in ϵ we have that $C = 1 + 2\epsilon$ and $T = 1 - 2\epsilon$, while $\frac{1}{\sinh(2kd)} = 2\epsilon$.

$$\left(\sin\xi + \frac{\alpha}{2}(1-2\epsilon) + 2\alpha\epsilon\sin^2\xi\right)$$
$$+\left(\frac{gk}{\omega^2}(1-2\epsilon) + \alpha^2 + \alpha(1-2\epsilon)\sin\xi\right)\mu + \frac{\alpha^2}{2}\sin\xi\ \mu^2 = 0$$

and

$$(\sin\xi - K) + (1 + \alpha(1+2\epsilon)\sin\xi)\,\mu + \frac{\alpha^2}{2}\sin\xi\ \mu^2 = 0$$

Still no good. :(

We could try replacing the $\sin^2\xi$ term with it's average $\frac{1}{2}$ and just ignoring the $\alpha\epsilon\sin\xi\ \mu$ terms (which have opposite sign), but this takes us almost back to the deep water limit, but with the $\frac{gk}{\omega^2}$ term multiplied by $(1-2\epsilon) = \tanh(kd)$.

$$\left(\sin\xi + \frac{\alpha}{2}\right) + \left(\frac{gk}{\omega^2}(1-2\epsilon) + \alpha^2 + \alpha\sin\xi\right)\mu + \frac{\alpha^2}{2}\sin\xi\ \mu^2 = 0$$

$$(\sin\xi - K) + (1 + \alpha\sin\xi)\,\mu + \frac{\alpha^2}{2}\sin\xi\ \mu^2 = 0$$

I guess it does give us an altered dispersion relation

$$\omega^2 = gk(1-2\epsilon) + (ka)^2 = gk\tanh(kd) + (ka)^2$$

which is a hybrid of the deep water $\omega^2 = gk+(ka)^2$ and small amplitude $\omega^2 = gk\tanh(kd)^2$ dispersion relations. Though it is not clear if this is closer to the truth, since we have truncated things in an uncontrolled way.

6 More Kinsman problems

§ 6.1 Kinsman section 3.4

Wave potential energy: $\mathtt{V}$

If we define the potential energy of the water to be zero when the sea is flat then the potential energy of a surface area dA with a disturbance η to the surface is

$$\begin{aligned} dU &= dm\ g\ z_{\text{cm}} \\ &= \rho\ dV\ \ g\ z_{\text{cm}} \\ &= \rho\ dV\ \ g\ \frac{\eta}{2} \\ &= \rho\ \eta\ dA\ g\ \frac{\eta}{2} \\ &= \frac{\rho g}{2}\eta^2 dA \end{aligned}$$

So the energy over an area A is

$$U = \int_A dA \frac{\rho g}{2}\eta^2$$

and the potential energy density is

$$\mathtt{V} = \frac{U}{A} = \frac{1}{A}\int_A dA\frac{\rho g}{2}\eta^2 = \frac{\rho g}{2}\frac{1}{A}\int_A dA\ \eta^2$$

But $\frac{1}{A}\int_A dA\ \eta^2$ is the mean of η^2 over the area so

$$\mathtt{V} = \frac{\rho g}{2}\langle\eta^2\rangle$$

Theorem: **Potential energy density**

$$\mathtt{V} = \frac{U}{A} = \tfrac{1}{2}\rho g\langle\eta^2\rangle$$

In the case that η is sinusoidal with amplitude a we know that $\langle\eta^2\rangle = \frac{1}{2}a^2$ and then

$$\mathtt{V} = \frac{\rho g a^2}{4}$$

Wave kinetic energy: $\mathtt{T}$

If we take the deep water limit we know that $\phi = \frac{a\omega}{k} e^{kz} \cos\xi$ and thus that the magnitude of the velocity in the water is constant $u = a\omega e^{kz}$ that does not change over time. We can then compute the total kinetic energy in a column of water from the bottom up to the surface as

$$\begin{aligned} T &= \int dT \\ &= \int \tfrac{1}{2} dm\ u^2 \\ &= \int \tfrac{1}{2}\rho dV\ u^2 \\ &= \rho \int_{-\infty}^{0} \tfrac{1}{2} A dz\ u^2 \\ &= \tfrac{1}{2} A\rho a^2 \omega^2 \int_{-\infty}^{0} dz\ e^{2kz} \\ &= \tfrac{1}{2} A\rho a^2 \omega^2 \frac{1}{2k} \\ &= A\frac{\rho g a^2}{4} \end{aligned}$$

Thus the kinetic energy density is

$$\mathtt{T} = \frac{T}{A} = \frac{\rho g a^2}{4}$$

The same as the potential energy density!

Theorem: **Mechanical energy density**
The total mechanical energy density is then

$$\mathtt{E} = \mathtt{T} + \mathtt{V} = \frac{\rho g a^2}{2} = \rho g \langle \eta^2 \rangle$$

Wave power

The wave power $\mathcal{P}$ through and area A with it's normal in the $\hat{x}$

direction is

$$\begin{aligned}
\mathcal{P} &= \int d\vec{F}\cdot\vec{u} \\
&= \int P\ d\vec{A}\cdot\vec{u} \\
&= \int P\ dA\ \hat{x}\cdot\vec{u} \\
&= \int dA\ P\ u_x
\end{aligned}$$

The time average power will be

$$\begin{aligned}
\langle\mathcal{P}\rangle_t &= \frac{1}{T}\int_0^T dt\ \mathcal{P} \\
&= \frac{1}{T}\int_0^T dt\int dA\ u_x P \\
&= \int dA\ \frac{1}{T}\int_0^T dt\ u_x P
\end{aligned}$$

We get the pressure from $\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+gz+\frac{P}{\rho}=0$ which along with $\phi=\frac{a\omega}{k}e^{kz}\cos\xi$ and $\frac{\omega^2}{k}=g$ gives

$$\begin{aligned}
P &= -\rho g a\ e^{kz}\sin\xi-\tfrac{1}{2}\rho a^2\omega^2e^{2kz}-\rho g z \\
u_x &= -a\omega e^{kz}\sin\xi
\end{aligned}$$

thus

$$\begin{aligned}
\frac{1}{T}\int_0^T dt\ u_x P &= \frac{1}{T}\int_0^T dt\ a\omega e^{kz}\sin\xi(\rho g a\ e^{kz}\sin\xi+\tfrac{1}{2}\rho a^2\omega^2e^{2kz}+\rho g z) \\
&= a\omega e^{kz}\rho g a\ e^{kz}\frac{1}{T}\int_0^T dt\ \sin^2\xi \\
&\quad + a\omega e^{kz}(\tfrac{1}{2}\rho a^2\omega^2e^{2kz}+\rho g z)\frac{1}{T}\int_0^T dt\ \sin\xi \\
&= a\omega e^{kz}\rho g a\ e^{kz}\ \frac{1}{2}+a\omega e^{kz}(\tfrac{1}{2}\rho a^2\omega^2e^{2kz}+\rho g z)\ 0 \\
&= \frac{\rho g a^2}{2}\omega e^{2kz} \\
&= \mathtt{E}\omega e^{2kz}
\end{aligned}$$

Now we can go back to getting the time average power.

$$\begin{aligned}\langle \mathcal{P} \rangle_t &= \int dA \; \frac{1}{T} \int_0^T dt \; u_x P \\ &= \int dA \; \mathtt{E} \; \omega e^{2kz} \\ &= \mathtt{E} \; \omega \int dA \; e^{2kz}\end{aligned}$$

Let us find the power through a strip w wide and going from the bottom up to the surface. Then we can write

$$\begin{aligned}\langle \mathcal{P} \rangle_t &= \mathtt{E} \; \omega \int dA \; e^{2kz} \\ &= \mathtt{E} \; \omega \int_0^w dy \int_{-\infty}^0 dz \; e^{2kz} \\ &= \mathtt{E} \; \omega \; w \int_{-\infty}^0 dz \; e^{2kz} \\ &= w \; \mathtt{E} \; \frac{\omega}{2k} \left[e^{2kz} \right]_{-\infty}^0 \\ &= w \; \mathtt{E} \; \frac{\omega}{2k} \\ &= w \; \mathtt{E} \; v_g\end{aligned}$$

and

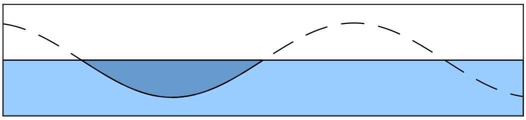
$$\frac{\langle \mathcal{P} \rangle_t}{w} = \mathtt{E} \; v_g$$

So we see that the energy of the wave moves with the group velocity not the phase velocity!

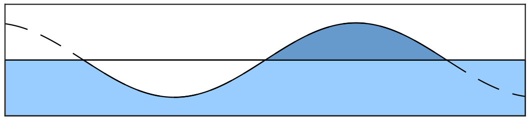
▷ Problem 6.1

Compute the change in the potential energy of the chunck of dark blue water shown in the diagram below, as if moves from what will become the trough of a wave to what becomes the peak of the wave. Show that this result is the same as the energy we derived in class (and Kinsman does in the text) for the potential energy.

before



after



▷ PROBLEM 6.2
Suppose that you have designed a panel that can take the incoming energy of the wave motion and generate power with it. Suppose that it has an efficiency of α, that is $\alpha = \frac{\text{power generated}}{\text{wave power}}$ the ratio of the power generated and the wave power striking the panel. This panel is placed vertically in the water with its top at the surface, and oriented so that the normal to the face is parallel to the velocity of the waves travel. The panel has a width w and a length ℓ down into the water, and a negligible thickness.
(a) Compute the wave power striking the panel, in the deep water limit.
(b) How does the power change with the amplitude of the wave? For the remainder of this problem assume an amplitude of 1 meter.
(c) Graph the wave power divided by the width w, versus the length ℓ for ℓ from 0 to 100 meters for periods of 5, 10, 15, and 20 seconds.
(d) Graph the wave power divided by panel area $w\ell$, versus the length ℓ for ℓ from 0 to 100 meters for periods of 5, 10, 15, and 20 seconds.
(e) If you were limited by the total area of panel would it be better to make a wide and shallow panel or a narrow and deep panel?
(f) Suppose that you have a panel that is one meter by one meter, what efficiency would you need to run an 80 watt laptop computer on a day with a swell of 0.5 meter amplitude?
(g) Compare this with the power generated from a solar panel of the same area. Sunlight has an intensity of about 1000 watts per square meter, but the sun is not always up and not always normal to the panel so the average power is reduced by a at least a factor of $\frac{1}{\pi}$. In addition solar panels are at best 37 % efficient in converting the solar radiation into electrical power.

§ 6.2 Kinsman chapter 4

▷ PROBLEM 6.3
Write a program `super(N,kbar,dk)` to compute the sum of small amplitude waves $\eta(x,t)$ described below, and graph the wave at sequential times in order to see the time evolution of the wave.

$$\eta = \sum_{n=0}^{N} a_n \cos(k_n x - \omega_n t)$$

with

$$\begin{aligned} k_n &= \bar{k} + \left(\frac{n}{N} - \frac{1}{2}\right) \delta k \\ \omega_n &= \sqrt{g k_n} \\ a_n &= 0.54 - 0.46 \cos(2\pi n/N) \end{aligned}$$

(a) Add a moving dot to the graph that has position $x = \frac{1}{2}\sqrt{g/\bar{k}}\ t$. Discuss the movement of this dot compared with the movement of the wave.
(b) Consider the following parameters

$$\begin{aligned} \bar{k} &= 0.01 \\ \delta k &= 0.0005 \end{aligned}$$

this will give you a range of periods from about 19.5s to 20.5s. Now see what happens as you increase the number of terms N.

Sum of a continuum of waves

In the previous problem we computed the sum

$$\eta = \frac{1}{N} \sum_{n=0}^{N} a_n \cos(k_n x - \omega_n t)$$

We add the scaling factor $1/N$ here to make the sum more invariant as we increase N.

Let us generalize this to an integral. First we note that the change from one k to the next is $dk = \frac{\delta k}{N}$, so we can write

$$\eta = \frac{1}{N} \sum_{n=0}^{N} a_n \cos(k_n x - \omega_n t) = \frac{1}{\delta k} \sum_{n=0}^{N} a_n \cos(k_n x - \omega_n t) dk$$

and

$$\begin{aligned} \eta &= \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} a_n \cos(k_n x - \omega_n t) \\ &= \lim_{dk \to 0} \frac{1}{\delta k} \sum_{n=0}^{N} a_n \cos(k_n x - \omega_n t) dk \\ &= \frac{1}{\delta k} \int a \cos(kx - \omega t) dk \end{aligned}$$

where a and ω are functions of k.

$$\eta = \frac{1}{\delta k} \int a \cos(kx - \omega t)\ dk = \mathtt{Re}\left[\int a e^{i(kx - \omega t)}\ \frac{dk}{\delta k}\right] = \mathtt{Re}[\psi]$$

Let us suppose $a(k)$ is negligibly small except for k near $\bar{k}$, that is $a(k) \approx 0$ for k such that $|k - \bar{k}| > \delta k$ for some small δk. In that case we can approximate

$$\omega(k) \approx \omega(\bar{k}) + \left.\frac{\partial \omega}{\partial k}\right|_{k=\bar{k}} (k - \bar{k})$$

Let us define $\bar{\omega} = \omega(\bar{k})$ and $v_g = \left.\frac{\partial \omega}{\partial k}\right|_{k=\bar{k}}$. Then we can write

$$\begin{aligned}
\psi &= \int_{\bar{k}-\delta k/2}^{\bar{k}+\delta k/2} a e^{i(kx-\omega t)} \frac{dk}{\delta k} \\
&= e^{i(\bar{k}x-\bar{\omega}t)} \int_{\bar{k}-\delta k/2}^{\bar{k}+\delta k/2} a e^{i((k-\bar{k})x-(\omega-\bar{\omega})t)} \frac{dk}{\delta k} \\
&= e^{i(\bar{k}x-\bar{\omega}t)} \int_{\bar{k}-\delta k/2}^{\bar{k}+\delta k/2} a e^{i((k-\bar{k})x-v_g(k-\bar{k})t)} \frac{dk}{\delta k} \\
&= e^{i(\bar{k}x-\bar{\omega}t)} \int_{\bar{k}-\delta k/2}^{\bar{k}+\delta k/2} a e^{i(k-\bar{k})(x-v_g t)} \frac{dk}{\delta k} \\
&= e^{i(\bar{k}x-\bar{\omega}t)} \int_{-1/2}^{1/2} a(\kappa) e^{i\kappa\xi} d\kappa \\
&= e^{i(\bar{k}x-\bar{\omega}t)} \sqrt{2\pi} A(\xi)
\end{aligned}$$

with $\kappa = \frac{k-\bar{k}}{\delta k}$ and $\xi = \delta k(x - v_g t)$, and where A is the inverse Fourier transform of a. Assuming that A is real then the real part of ψ is

$$\eta = \cos(\bar{k}x - \bar{\omega}t)\sqrt{2\pi} A(\xi)$$

But since $\xi = \delta k(x - v_g t)$ we see that the envelope $A(\xi)$ moves with the group velocity, and that the oscillation $\cos(\bar{k}x - \bar{\omega}t)$ is modulated by the envelope.

As a specific example let us consider the previous homework problem again.

$$\begin{aligned}
k_n &= \bar{k} + \left(\frac{n}{N} - \frac{1}{2}\right)\delta k \;\longrightarrow\; \kappa = \frac{k-\bar{k}}{\delta k} = \frac{n}{N} - \frac{1}{2} \\
\longrightarrow\; \frac{n}{N} &= \kappa + \tfrac{1}{2} \;\longrightarrow\; 2\pi\frac{n}{N} = 2\pi\kappa + \pi \\
\longrightarrow\; a &= 0.54 - 0.46\cos(2\pi n/N) \\
&= 0.54 - 0.46\cos(2\pi\kappa + \pi) \\
&= 0.54 + 0.46\cos(2\pi\kappa) \\
&= 0.54 + 0.23e^{i2\pi\kappa} + 0.23e^{-i2\pi\kappa}
\end{aligned}$$

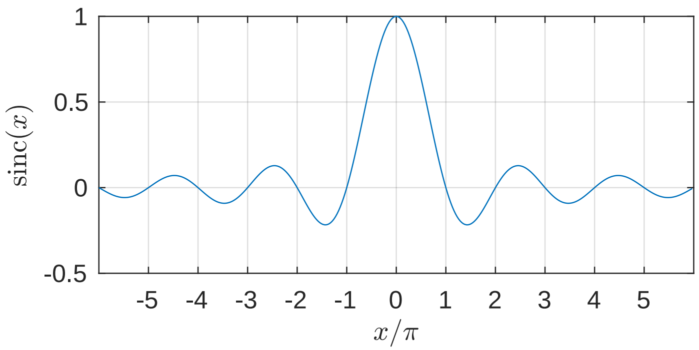
and so

$$\int a(\kappa)e^{i\kappa\xi}d\kappa = \int_{-1/2}^{1/2} \left[0.54 + 0.23e^{i2\pi\kappa} + 0.23e^{-i2\pi\kappa}\right] e^{i\kappa\xi}d\kappa$$
$$= \int_{-1/2}^{1/2} \left[0.54e^{i\kappa\xi} + 0.23e^{i\kappa(\xi+2\pi)} + 0.23e^{i\kappa(\xi-2\pi)}\right] d\kappa$$

These are all of the form

$$\int_{-1/2}^{1/2} e^{i\kappa b}d\kappa = \frac{e^{i\frac{b}{2}} - e^{-i\frac{b}{2}}}{ib} = \frac{2i\sin\frac{b}{2}}{ib} = \frac{\sin\frac{b}{2}}{\frac{b}{2}} = \mathrm{sinc}\left(\frac{b}{2}\right)$$

The sinc function is graphed below.

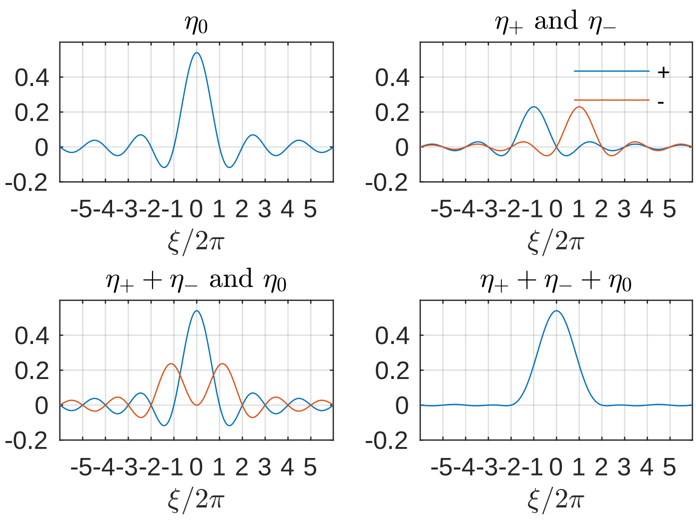


$$\int a(\kappa)e^{i\kappa\xi}d\kappa = 0.54 \text{ sinc}\left(\frac{\xi}{2}\right)$$
$$+ 0.23 \text{ sinc}\left(\frac{\xi+2\pi}{2}\right) + 0.23 \text{ sinc}\left(\frac{\xi-2\pi}{2}\right)$$
$$= 0.54 \text{ sinc}\left(\frac{\xi}{2}\right)$$
$$+ 0.23 \text{ sinc}\left(\frac{\xi}{2}+\pi\right) + 0.23 \text{ sinc}\left(\frac{\xi}{2}-\pi\right)$$

$$\equiv \eta_0(\xi) + \eta_+(\xi) + \eta_-(\xi)$$

Now we can get the final waveform η.

$$\begin{aligned}\eta &= \mathtt{Re}\left[\int a e^{i(kx-\omega t)}\ dk\right] = \mathtt{Re}\left[e^{i(\bar{k}x-\bar{\omega}t)}\int a(\kappa)e^{i\kappa\xi}d\kappa\right]\\ &= \cos(\bar{k}x-\bar{\omega}t)\int a(\kappa)e^{i\kappa\xi}d\kappa\\ &= \cos(\bar{k}x-\bar{\omega}t)\Bigg[0.54\ \mathrm{sinc}\left(\frac{\xi}{2}\right)\\ &\quad + 0.23\ \mathrm{sinc}\left(\frac{\xi}{2}+\pi\right) + 0.23\ \mathrm{sinc}\left(\frac{\xi}{2}-\pi\right)\Bigg]\\ &= \cos(\bar{k}x-\bar{\omega}t)\Big[\eta_0(\xi)+\eta_+(\xi)+\eta_-(\xi)\Big]\end{aligned}$$



▷ Problem 6.4

Consider the wave

$$\eta(x,t) = \frac{1}{\delta k}\int a(k)\cos(kx-\omega t)\ dk$$

with the wave number amplitude

$$a(k) = \begin{cases} a_0 & \text{if } |k-\bar{k}| < \delta k/2 \\ 0 & \text{otherwise}\end{cases}$$

Graph η versus $\xi = \delta k(x - v_g t)$.

▷ Problem 6.5

Consider the wave

$$\eta(x,t) = \int_{-\infty}^{\infty} a(k)\cos(kx - \omega t)\; dk$$

with a gaussian wave number amplitude

$$a(k) = \frac{1}{\sqrt{2\pi\sigma_k^2}} e^{-\frac{1}{2}(\frac{k-\bar{k}}{\sigma_k})^2}$$

Graph η versus $\xi = \sigma_k(x - v_g t)$. Show that the envelope of $\eta(\xi)$ is a gaussian too.

§ 6.3 Holthuijsen section 3.1 through 3.4

▷ PROBLEM 6.6
Read sections 3.1 through 3.4.

▷ PROBLEM 6.7
Take the buoy data provided and compute the wave height H and period T_0 for all of the waves in the time series. Break the data into half hour long segments.
(a) Plot the histogram of wave heights for each half hour. Mark H_{rms}, $\bar{H}$, $H_{1/3}$, and $H_{1/10}$ on the histogram.
(b) Plot the histogram of periods for each half hour. Mark T_{rms}, $\bar{T}_0$, $T_{1/3}$, and $T_{1/10}$ on the histogram.

§ 6.4 Holthuijsen section 3.5

Theorem: **Discrete Fourier Transform**
Suppose you have a signal $\eta(t)$ sampled at times $t_k = \frac{k}{f_s}$ where f_s is the sample rate, so that we have the sequence of values $\eta_k = \eta(t_k)$. The sequences H_n is defined by the following.

$$\eta_k = \sum_{n=0}^{N-1} H_n e^{2\pi nk/N}$$

$$H_n = \sum_{k=0}^{N-1} \eta_k e^{-2\pi nk/N} = \sum_{k=1-N/2}^{N/2} \eta_k e^{-2\pi nk/N}$$

The sequence H_n is called the *Discrete Fourier Transform* (DFT) of the sequence η_k, and is associated with the frequencies $f_n = n\frac{f_s}{N}$ since

$$f_n t_k = n\frac{f_s}{N}\frac{k}{f_s} = \frac{nk}{N}$$

Theorem: **Parceval's Theorem**
If H_n is the DFT of the sequence η_k then

$$\sum_{k=0}^{N-1} |\eta_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |H_n|^2$$

We can use the above to write the wave power in terms of H_n. Recall that the wave power was proportional to $\langle \eta^2 \rangle$ the mean value of η^2. Suppose that we have recorded N samples $\eta_k = \eta(t_k)$ with a sample rate of f_s, that is $t_k = k/f_s$.

$$\begin{aligned}
\langle \eta^2 \rangle &= \frac{1}{N} \sum_{k=0}^{N-1} \eta_k^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\eta_k|^2 \\
&= \frac{1}{N}\frac{1}{N} \sum_{n=0}^{N-1} |H_n|^2 \quad \text{by Parceval's theorem} \\
&= \sum_{n=0}^{N-1} \left| \frac{H_n}{N} \right|^2 \\
&= 2 \sum_{n=1}^{N/2-1} \left| \frac{H_n}{N} \right|^2 + \left| \frac{H_{N/2}}{N} \right|^2 \quad \text{used } H_0 = 0 \text{ since } \langle \eta \rangle = 0 \\
&= 2 \sum_{n=1}^{N/2-1} \left| \frac{H_n}{N} \right|^2 \quad \text{assuming } H_{N/2} = 0 \\
&= \sum_{n=1}^{N/2-1} \frac{2\,|H_n|^2}{N^2 \Delta f} \Delta f \\
&= \sum_{n=1}^{N/2-1} \frac{2\,|H_n|^2}{N f_s} \Delta f
\end{aligned}$$

Because of this result we are lead to the following definition of the *variance density spectrum* $E(f_n)$.

$$E(f_n) = \left\lceil \frac{2\,|H_n|^2}{N f_s} \right\rfloor = \frac{2}{N f_s} \left\lceil |H_n|^2 \right\rfloor$$

where the symbol $\lceil \texttt{blob} \rfloor$ is a notation for the *ensemble mean* or *expectation value* of `blob`. In this case the ensemble is a collection of sequences h_k and so the DFT gives an ensemble of H_n, so $\lceil |H_n|^2 \rfloor$ is notation for the mean of $|H_n|^2$ over this ensemble. The idea is that $E(f)$ should be independent of our sample rate and the number of samples taken, that is independent of the method of measurement.

The textbook uses the notation $E\{\texttt{blob}\}$ for the expectation value of `blob` which seems confusing since E already has a meaning. Thus were will do the following

$$E\{\texttt{blob}\} \Longrightarrow \lceil\texttt{blob}\rfloor$$

I hope that this makes things more clear.

We can also express the variance density spectrum in terms of the amplitudes a_n of the cosine representation of the surface

$$\begin{aligned}
\eta_k &= \sum_{n=1}^{N/2-1} a_n \cos(2\pi f_n t_k + \alpha_n) \\
&= \sum_{n=1}^{N/2-1} a_n \cos(2\pi nk/N + \alpha_n) \\
&= \sum_{n=1}^{\frac{N}{2}-1} \frac{a_n}{2} e^{i2\pi nk/N} e^{i\alpha_n} + \sum_{n=1}^{\frac{N}{2}-1} \frac{a_n}{2} e^{-i2\pi nk/N} e^{-i\alpha_n} \\
&= \sum_{n=1}^{\frac{N}{2}-1} b_n e^{i2\pi nk/N} + \sum_{n=1}^{\frac{N}{2}-1} b_n^* e^{-i2\pi nk/N}
\end{aligned}$$

with $b_n = \frac{a_n}{2} e^{i\alpha_n}$. These b_n are just defined for the positive values of n. Let us define $b_{-n} = b_n^*$ then we can write

$$\begin{aligned}
\eta_k &= \sum_{n=1}^{\frac{N}{2}-1} b_n e^{i2\pi nk/N} + \sum_{n=1}^{\frac{N}{2}-1} b_{-n} e^{i2\pi(-n)k/N} \\
&= \sum_{n=1}^{\frac{N}{2}-1} b_n e^{i2\pi nk/N} + \sum_{n=-1}^{-\frac{N}{2}+1} b_n e^{i2\pi nk/N} \\
&= \sum_{n=1}^{\frac{N}{2}-1} b_n e^{i2\pi nk/N} + \sum_{n=-1}^{1-\frac{N}{2}} b_n e^{i2\pi nk/N} \\
&= \sum_{n=1-\frac{N}{2}}^{N/2-1} b_n e^{i2\pi nk/N}
\end{aligned}$$

with $b_0 = 0$.

Comparing this with

$$\eta_k = \frac{1}{N} \sum_{k=1-N/2}^{N/2} H_n e^{2\pi nk/N}$$

We see that $b_n = \frac{1}{N} H_n$ Thus $\frac{a_n}{2} e^{i\alpha_n} = \frac{1}{N} H_n$ and so

$$H_n = N \frac{a_n}{2} e^{i\alpha_n}$$

and

$$|H_n| = N \frac{a_n}{2}$$

and so

$$E(f_n) = \left\lceil \frac{2\,|H_n|^2}{N f_s} \right\rceil = \left\lceil \frac{N a_n^2}{2 f_s} \right\rceil = \left\lceil \frac{a_n^2}{2\Delta f} \right\rceil$$

▷ PROBLEM 6.8
Using the same buoy data from March 2.
(a) Compute and plot $E(f_n)$ for each half hour of data.
(b) Compute the mean of η^2 directly from the data for each half hour. Also compute $\sum_n E(f_n)\Delta f$ for each half hour. Graph both versus the "group" number (which half hour). Are they the same?

§ 6.5 Characterizing the variation in $E(f)$: LH ??

We have defined $E(f_n)$ in terms of the ensemble average of $\frac{a_n^2}{2\Delta f}$. would like to understand how the individual values of $\frac{a_n^2}{2\Delta f}$ vary about the ensemble average. This section will investigate this distribution.

Consider one component of the wave $\eta_n = a_n \cos(2\pi f_n t + \phi_n)$ and think that we could have instead written

$$\eta_n = A_x \cos(2\pi f_n t) + A_y \sin(2\pi f_n t)$$

There is reason to believe that since the sin and cos function are independent that the coefficients A_x and A_y will be independent random variables with a gaussian distribution. Let us assume that this is so and see what happens. Thus suppose that the probability density for A_x is

$$p(A_x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-A_x^2/2\sigma^2} \qquad \text{AND} \qquad p(A_y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-A_y^2/2\sigma^2}$$

where σ is the standard deviation of the distribution for A_x and A_y both. So the joint probability for getting the pair of numbers (A_x, A_y) is

$$p(A_x, A_y) = p(A_x)p(A_y) = \frac{1}{2\pi\sigma^2} e^{-(A_x^2 + A_y^2)/2\sigma^2}$$

Now if think of the point (A_x, A_y) in a 2-D plane then we could just as well have written the point in terms polar coordinates (A, θ), and where $A_x = A\cos\theta$ and $A_y = A\sin\theta$ and thus

$$\eta_n = A\cos\theta\cos(2\pi f_n t) + A\sin\theta\sin(2\pi f_n t) = A\cos(2\pi f_n t - \theta)$$

Comparing this with $\eta_n = a_n \cos(2\pi f_n t + \phi_n)$, we see that our $A = a_n$ and $-\theta = \phi_n$ and the distribution of (a_n, ϕ_n) is the distribution of $(A, -\theta)$. The first thing to notice is that the probability density does not depend on θ, and thus it does not depend on ϕ_n.

$$p(A_x, A_y) = \frac{1}{2\pi\sigma^2} e^{-(A_x^2+A_y^2)/2\sigma^2} = \frac{1}{2\pi\sigma^2} e^{-A^2/2\sigma^2} = \frac{1}{2\pi\sigma^2} e^{-a_n^2/2\sigma^2}$$

We can take our infinitesimal area element in the (A_x, A_y) plane as

$$dA_x\ dA_y = 2\pi A\ dA = 2\pi a_n\ da_n$$

Thus

$$p(A_x, A_y) dA_x\ dA_y = \frac{1}{2\pi\sigma^2} e^{-a_n^2/2\sigma^2} 2\pi a_n\ da_n = \frac{1}{\sigma^2} a_N e^{-a_n^2/2\sigma^2} da_n$$

Thus we see that the distribution for A and thus a_n is

$$p(a_n) = \frac{1}{\sigma_n^2} a_n e^{-a_n^2/2\sigma_n^2}$$

This distribution is a *Rayleigh Distribution.*

Now since $E(f_n) = \left\lceil \frac{a_n^2}{2\Delta f} \right\rfloor$ what we would really like is the distribution of $\varepsilon_n = \frac{a_n^2}{2\Delta f}$. First we note that $d\varepsilon_n = \frac{1}{\Delta f} a_n da_n$, Thus

$$p(a_n) da_n = \frac{1}{\sigma_n^2} a_n e^{-a_n^2/2\sigma_n^2} da_n = \frac{\Delta f}{\sigma_n^2} e^{-\Delta f \varepsilon_n/\sigma_n^2} d\varepsilon_n$$

So evidently the distribution in ε_n is

$$p(\varepsilon_n) = \frac{\Delta f}{\sigma_n^2} e^{-\Delta f \varepsilon_n/\sigma_n^2}$$

which is an *exponential distribution.*

▷ Problem 6.9

Show that the ensemble mean of the Rayleigh distribution is $\lceil a_n \rfloor = \sqrt{\frac{\pi}{2}} \sigma_n$, and that the Rayleigh distribution can be written in terms of the mean $\mu = \lceil a_n \rfloor$ as

$$p(a_n) = \frac{\pi}{2\mu^2} a_n e^{-\pi a_n^2/4\mu^2}$$

▷ Problem 6.10

Show that $E(f_n) = \sigma_n^2/\Delta f$, and that the distribution of ε_n can be written as

$$p(\varepsilon_n) = \frac{1}{E_n} e^{-\varepsilon_n/E_n}$$

where $E_n = E(f_n)$

▷ Problem 6.11

Notice that if we let $u = \frac{\varepsilon_n}{E_n}$ then $p(\varepsilon_n) d\varepsilon_n = e^{-u} du$, thus u has the probability density $p(u) = e^{-u}$. Using the March 2 buoy data compute the ε_n for n such that $f_n = 0.07$Hz (this is near the peak). Make an estimate of the mean E_n for each 30 minute section and use this to compute u. Pool all the data together and make a histogram of the data. Set the histogram so that is displays the probability density. On top of the histogram plot e^{-u}.

▷ Problem 6.12
Consider an exponential distribution $p(x) = \frac{1}{\mu}e^{-x/\mu}$ where μ is the expectation value of x, and where x goes from zero to ∞. Find the standard deviation of x.

Theorem: **Variance theorem**
Suppose that we have a set of random variables x_n with means μ_n and standard deviations σ_n. Then $x = \sum_n x_n$ is also a random variable and its mean is $\mu = \sum_n \mu_n$ and the standard deviation σ is $\sigma = \sqrt{\sum_n \sigma_n^2}$.

▷ Problem 6.13
Using the theorem above show that if you take N samples of a random variable x with mean μ and standard deviation σ and compute the mean of these N random variables, $M = \frac{1}{N}\sum_n x_n$, then $\lceil M \rfloor = \mu$ and show that the standard deviation of M is $\sigma/\sqrt{N}$. Recall that for one half hour of buoy data we took the mean of 18 measurements to compute E_n. What is the uncertainty in this mean?

▷ Problem 6.14
Let us investigate the above result more numerically, since the data we can get from the buoy's is not really enough to provide a clear picture of the limit. Let us suppose that x is exponentially distributed with a mean of a then the probability density is $p(x) = \frac{1}{a}e^{-x/a}$. We can generate an `N` by `M` array of such random numbers by the code

```
x = -a*log(rand(N,M))
```

You can then by taking the mean of the columns of this array create a vector of `M` values. These values are the mean of `N` of the random values of x. Let us refer to these means as $\langle x \rangle_N$. Do this for $N = 18$ and $M = 100000$.
(a) Plot a histogram of these means.
(b) Compute the mean and standard deviation of the M values of $\langle x \rangle_N$.
(c) Do the numerically computed mean and standard deviation agree with the theoretical values?

7 Wave Statistics

§ 7.1 Holthuijsen Chapter 4

▷ PROBLEM 7.1

Check the applicability of equations 4.2.1 and 4.2.2 for the March 2 buoy data that you have.

Hint for equation 4.2.1: If we scale η by $\sqrt{m_0}$, that is let $u = \eta/\sqrt{m_0}$ be the scaled η, then the theoretical prediction for the pdf of u is that $p(u) = \sqrt{\frac{1}{2\pi}}e^{-u^2/2}$, which is independent of m_0. This implies that the probability density of the scaled η will not shift with the changing sea state. Thus if you compute m_0 for each half hour and compute u for this half hour then you can pool all the u's for the full day and create one histogram for the entire day.

Hint for equation 4.2.2: We want to verify equation 4.2.2 which is,

$$\bar{T}_\eta = \sqrt{\frac{m_0}{m_2}}e^{\eta^2/2m_0}$$

which can be recast as

$$2m_0 \log\left(\bar{T}_\eta\sqrt{\frac{m_2}{m_0}}\right) = \eta^2$$

or

$$m_0 \log\left(\bar{T}_\eta^2\,\frac{m_2}{m_0}\right) = \eta^2$$

So compute $\bar{T}_\eta$, m_0, m_2 for each half hour segment and then compute the $\Upsilon(\eta) = m_0 \log\left(\bar{T}_\eta^2\,\frac{m_2}{m_0}\right)$ for each half hour segment . Do this for a range of values of η from 0 to 1.6 and then pool all the data for all of the half hour segments and see if $\Upsilon(\eta)$ is equal to η^2.

▷ PROBLEM 7.2

Use the March 2 buoy data to see if H_{m_0}, as defined in equation 4.2.24, gives a good estimate of $H_{1/3}$. Recall that you already have a program that computes $H_{1/3}$. Make a graph like figure 4.11 from the textbook.

§ 7.2 Holthuijsen Chapter 6

Theorem: **Pierson-Moskowitz Spectral Density**

The following function has been used to model the spectral density.

$$E_{\mathrm{PM}}(f) = \alpha_{\mathrm{PM}}\frac{g^2}{(2\pi)^4 f^5}e^{-\frac{5}{4}\left(\frac{f_{\mathrm{PM}}}{f}\right)^4}$$

▷ PROBLEM 7.3

Write α_{PM} in terms of $\lceil \eta^2 \rfloor$. Rewrite the Pierson-Moskowitz spectral density in with this expression of α_{PM} in terms of $\lceil \eta^2 \rfloor$. There are hints for this problem.

▷ PROBLEM 7.4

Let f_p be the value of the frequency that maximizes the Pierson-Moskowitz spectral density. What is f_p? Let $u = f/f_p$ and write the spectral density in terms of u.

▷ PROBLEM 7.5

Check to see if the Pierson-Moskowitz spectral density fits the buoy data for March 2 and March 7. The results of the previous two problems will help you estimate the constants α_{PM} and f_{PM} from the observed f_p and $\lceil \eta^2 \rfloor$.

▷ PROBLEM 7.6

Check to see if the JONSWAP spectral density fits the buoy data for March 2, 6 and 7.

▷ PROBLEM 7.7

Let there be two waves with different wave vectors, $\vec{k}_1$ and $\vec{k}_2$ on the same surface.

$$\psi = \tfrac{1}{2}\cos(\vec{k}_1 \cdot \vec{r} - \omega_1 t) + \tfrac{1}{2}\cos(\vec{k}_2 \cdot \vec{r} - \omega_2 t)$$

We will assume the deep water limit so that $\omega_1^2 = gk_1$ and $\omega_2^2 = gk_2$. We can rewrite the above as

$$\psi = \cos(\vec{k}_- \cdot \vec{r} - \omega_- t)\cos(\vec{k}_+ \cdot \vec{r} - \omega_+ t)$$

with

$$\vec{k}_\pm = \frac{\vec{k}_1 \pm \vec{k}_2}{2} \qquad \text{AND} \qquad \omega_\pm = \frac{\omega_1 \pm \omega_2}{2}$$

Is it possible for $\omega_+^2 = gk_+$?

▷ PROBLEM 7.8

Now consider four wave mixing. In this case we have the resonance condition that

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4 \qquad \text{AND} \qquad \omega_1 + \omega_2 = \omega_3 + \omega_4$$

for the energy to be transferred from one component to another. For each of the components $\omega_n^2 = gk_n$. Define $z = \frac{\omega_2}{\omega_1}$, $x = \frac{\omega_3}{\omega_1}$, and $y = \frac{\omega_4}{\omega_1}$, and let α be the cosine of the angle between $\vec{k}_1$ and $\vec{k}_2$ and let the β be the cosine of the angle between $\vec{k}_3$ and $\vec{k}_4$. Write out the two conditions

$$\left(\vec{k}_1 + \vec{k}_2\right) \cdot \left(\vec{k}_1 + \vec{k}_2\right) = \left(\vec{k}_3 + \vec{k}_4\right) \cdot \left(\vec{k}_3 + \vec{k}_4\right) \quad \text{AND} \quad \omega_1 + \omega_2 = \omega_3 + \omega_4$$

in terms of x, y, z, α, β. What is dimension of the space of solutions to these two conditions. For example the dimension of the space defined by $x^2 + y^2 + z^2 = 1$ is 2, since it is a surface in a 3-D space.

8 Random Stuff

§ 8.1 Hyperbolic Functions

Definitions

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}$$
$$\sinh x \equiv \frac{e^x - e^{-x}}{2}$$
$$\tanh x \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$
$$\coth x \equiv \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\cosh x}{\sinh x}$$

Identities

$$\cosh^2 x = \tfrac{1}{2}[\cosh(2x) + 1]$$
$$\sinh^2 x = \tfrac{1}{2}[\cosh(2x) - 1]$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh^2 x + \sinh^2 x = \cosh(2x)$$
$$\cosh x \;\; \sinh x = \tfrac{1}{2} \sinh 2x$$

Derivatives

$$\frac{d}{dx} \cosh x = \sinh x$$
$$\frac{d}{dx} \sinh x = \cosh x$$
$$\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x$$

Expansions

For $|x| < 1$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
$$\tanh(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \mathcal{O}\left[x^7\right]$$

For $d > 0$ and where we have defined $\epsilon = e^{-2d}$.

$$\tanh(d) = 1 - 2\epsilon + 2\epsilon^2 - 2\epsilon^3 + 2\epsilon^4 - 2\epsilon^5 + \cdots$$

$$\coth(d) = 1 + 2\epsilon + 2\epsilon^2 + 2\epsilon^3 + 2\epsilon^4 + 2\epsilon^5 + \cdots$$

$$\frac{\cosh(d+z)}{\sinh(d)} = (e^z + \epsilon\ e^{-z})(1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots)$$

For $z < 1$

$$\frac{\cosh(d+z)}{\sinh(d)} = \coth(d) + z + \coth(d)\frac{z^2}{2!} + \frac{z^3}{3!} + \coth(d)\frac{z^4}{4!} + \cdots$$

$$\frac{\sinh(d+z)}{\sinh(d)} = 1 + \coth(d)\ z + \frac{z^2}{2!} + \coth(d)\frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

A Hints

2.1 Use the definition of pressure. 1.6×10^5Pa, 117.0×10^5Pa

2.2 Compare the pressure created by the elephant with the pressure created by your lungs. Yes

2.3 The pressure in your lungs will be the same as the pressure in the air above the water, and thus it will be the same as the pressure in the water at the surface. 5.2m

2.4 What is the acceleration of the water? Draw a free body diagram for the blob of water. Note that the force on the top and bottom is due to the pressure on the top and bottom.

2.5 The net force is due to pressure is the pressure difference times the area. 2.23×10^5N

2.6 Use $\Delta P = -\rho g \Delta y$. -1.48×10^4Pa, -1.18×10^7Pa

2.7 Use $\Delta P = -\rho g \Delta y$. The difference in pressure between the outside and inside of the box will be the same as the difference in pressure between the bottom of the sea and the surface of the sea.
4.0×10^4N, 6.0×10^4N, 1.2×10^5N

2.8 Use $\Delta P = -\rho g \Delta y$. 490Pa

2.9 The volume rate of flow is $\frac{dV}{dt} = Av$. 2.2m^3/s, 0.68m

2.10 Use projectile motion to find the velocity at the nozzle from the trajectory of the water. Find the volume rate of flow from this velocity. From the volume rate of flow find the time. 6.3 hours

2.11 The lift is due to the pressure difference between the upper and lower surfaces of the wing. Find the pressure difference from this. Use Bernoulli's equation to relate this pressure difference to the speed. $302.6\frac{m}{s}$

2.12 Use the equation of continuity and Bernoulli's equation. 1.45×10^{-3}m^3/s

2.13 The lift on each "wing" is perpendicular it's surface.

2.14 Use Bernoulli's equation. 6.0×10^4N

2.15 Use the continuity equation and Bernoulli's equation. 480Pa

2.16 Ask Bernoulli for help.

3.5 If H_n is the amplitude of the the FFT for frequency f_n then the power for f_n is proportional to $|H_n|^2$.

3.6 Make a grid of locations (x, y) on the sea surface and for each case a corresponding z value for each surface location. Compute the range for each point $r = \sqrt{(h-z)^2 + x^2 + y^2}$ and then collect them into range bins. The total power for a range bin is then the sum of the gain values for all of the points who's range falls in that bin.

4.1 The position of the bug will be $\vec{r} = a\cos\omega t\ \hat{x} + a\sin\omega t\ \hat{y}$. Find the velocity of the bug. Write the velocity of the bug in terms of its position x and y. The velocity of the bug at that location is $\vec{u}(x, y)$.

4.2 Expand both sides using the notation from this section.

4.3 Express both sides using the new notation.

4.5 Plug and chug.

4.6 In cylindrical coordinates the gradient is $\nabla P = \frac{\partial P}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial P}{\partial \theta}\hat{\theta} + \frac{\partial P}{\partial z}\hat{z}$.

4.8 After dealing with the rigid boundary condition use the dynamic BC next.

4.9 Use the definition of ϕ and compute, using the know properties of ϕ_1 and ϕ_2.

4.10 Just compute!

4.11 Plug into the Dynamic and Kinematic BC's and see what you get.

4.12 Dont forget that $\frac{\partial}{\partial k}\left[\omega^2\right] = 2\omega\frac{\partial \omega}{\partial k}$.

4.14 Use the expression for the group and phase velocities from before.

4.15 We want to integrate the differential equation.

$$\frac{d\vec{r}}{dt} = \vec{u}(\vec{r}(t), t)$$

for the given function

$$\vec{u}(\vec{r}, t) = a\omega e^{kr_z}\left[-\sin(kr_x - \omega t)\hat{x} + \cos(kr_x - \omega t)\hat{z}\right]$$

The Euler's method is to discretize this to the following recursion relation.

$$\vec{r}_{n+1} = \vec{r}_n + \vec{u}(\vec{r}_n, t_n)\Delta t$$

Since the motion is nearly circular you may want to use a higher order approach. For example the second order Runge-Kutta would be

$$\vec{r}_{\mathrm{mid}} = \vec{r}_n + \vec{u}(\vec{r}_n, t_n)\frac{\Delta t}{2}$$

$$t_{\mathrm{mid}} = t_n + \frac{\Delta t}{2}$$

$$\vec{r}_{n+1} = \vec{r}_n + \vec{u}(\vec{r}_{\mathrm{mid}}, t_{\mathrm{mid}})\Delta t$$

It will be helpful to scale the problem in order to get a more general solution. We can for example reduce the problem to one scale parameter $\alpha \equiv ka$ instead of the two parameters wavelength, and wave amplitude. This can be done as follows. We notice that

$$\frac{\vec{u}}{a\omega} = \frac{1}{a\omega}\frac{d\vec{r}}{dt} = \frac{d}{d[\omega t]}\left[\frac{\vec{r}}{a}\right]$$

So with defining a scaled time $\theta = \omega t$ and a scaled position $\vec{\beta} = \frac{\vec{r}}{a}$ we have that

$$\begin{aligned}\frac{d\vec{\beta}}{d\theta} = \frac{\vec{u}}{a\omega} &= e^{kr_z}\left[-\sin(kr_x - \theta)\hat{x} + \cos(kr_x - \theta)\hat{z}\right] \\ &= e^{\alpha\beta_z}\left[-\sin(\alpha\beta_x - \theta)\hat{x} + \cos(\alpha\beta_x - \theta)\hat{z}\right]\end{aligned}$$

This differential equation depends on the single scale parameter α.

The drift velocity v_d will be the ratio of the horizontal distance the water travels in one cycle $\Delta r_x = a\Delta\beta_x$ and the period T.

$$\frac{v_d}{v_p} = \frac{\Delta r_x/T}{\lambda/T} = \frac{\Delta r_x}{\lambda} = \frac{a\Delta\beta_x}{\lambda} = \frac{ka\Delta\beta_x}{k\lambda} = \frac{\alpha\Delta\beta_x}{2\pi}$$

4.16 Use $\frac{\cosh(kz+kd)}{\sinh(kd)} \approx e^{kz}$ and $\frac{\sinh(kz+kd)}{\sinh(kd)} \approx e^{kz}$.

4.17 The restriction $\omega^2 = gk\tanh(kd)$ is important. Use it to replace $\frac{\omega^2}{gk}$ with $\tanh(kd)$ in the dynamic BC.

6.1 In general to compute the gravitational potential energy of a distributed object we take the volume integral over the extent of the object

$$U = \int dm\ g\ z = \int \rho\ dV\ g\ z = \rho g \int dV\ z$$

where z is the vertical coordinate of the volume element dV.

6.2 To Be Done

6.3 To Be Done

6.4 Follow the example in the notes.

6.5 To Be Done

6.6 Do it!

6.8 To Be Done

6.10 recall that $E(f_n) = \lceil \varepsilon_n \rfloor$.

6.12 Start by computing the expectation value of x^2.

6.13 Recall that if $y = ax$ with x and y random variables and a a constant, then $\lceil y \rfloor = a\lceil x \rfloor$ and $\sigma_y = a\sigma_x$.

7.3 Recall that $m_0 = \lceil \eta^2 \rfloor$, and recall the definition of m_0. The substitution $u = -\frac{5}{4}\frac{f_{\mathrm{PM}}^4}{f^4}$ will make the integral trivial.

7.4 Since the log function is a monotonically increasing function, the maximum $\log[E(f)]$ is occurs for the same frequency as the maximum of $E(f)$.

7.7 To Be Done

7.8 To Be Done