The Sum of Two Sinusoids

Here we will derive a mathematical result that is useful for understanding the superposition of waves. First we note two trig identities that we will need.

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

and

$$\cos^2 a + \sin^2 a = 1$$

With these we can rewrite our statement of the superposition principle for sinusoidal waves

$$A\cos(\phi - \omega t) = A_1 \cos(\phi_1 - \omega t) + A_2 \cos(\phi_2 - \omega t)$$

in a way that is more useful.

$$\begin{aligned}
A\cos(\phi - \omega t) &= A_1 \cos(\phi_1 - \omega t) + A_2 \cos(\phi_2 - \omega t) \\
A(\cos\phi\cos\omega t + \sin\phi\sin\omega t) &= A_1(\cos\phi_1\cos\omega t + \sin\phi_1\sin\omega t) + A_2(\cos\phi_2\cos\omega t + \sin\phi_2\sin\omega t) \\
0 &= (A_1\cos\phi_1 + A_2\cos\phi_2 - A\cos\phi)\cos\omega t \\
&\quad + (A_1\sin\phi_1 + A_2\sin\phi_2 - A\sin\phi)\sin\omega t
\end{aligned}$$

The only way for this to be true for a times t is if both

$$(A_1\cos\phi_1 + A_2\cos\phi_2 - A\cos\phi) = 0 \qquad \text{and} \qquad (A_1\sin\phi_1 + A_2\sin\phi_2 - A\sin\phi) = 0.$$

These can be rewritten as

$$A\cos\phi = A_1\cos\phi_1 + A_2\cos\phi_2 \qquad \text{and} \qquad A\sin\phi = A_1\sin\phi_1 + A_2\sin\phi_2.$$

The ratio of these two equations give us

$$\tan\phi = \frac{A\sin\phi}{A\cos\phi} = \frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}$$

whereas adding the squares of the two equations gives us

$$\begin{aligned}
(A\cos\phi)^2 + (A\sin\phi)^2 &= (A_1\cos\phi_1 + A_2\cos\phi_2)^2 + (A_1\sin\phi_1 + A_2\sin\phi_2)^2 \\
A^2\cos^2\phi + A^2\sin^2\phi &= (A_1^2\cos^2\phi_1 + 2A_1A_2\cos\phi_1\cos\phi_2 + A_2^2\cos^2\phi_2) \\
&\quad + (A_1^2\sin^2\phi_1 + 2A_1A_2\sin\phi_1\sin\phi_2 + A_2^2\sin^2\phi_2) \\
A^2(\cos^2\phi + \sin^2\phi) &= A_1^2(\cos^2\phi_1 + \sin^2\phi_1) + A_2^2(\cos^2\phi_2 + \sin^2\phi_2) + 2A_1A_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) \\
A^2 &= A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)
\end{aligned}$$

This last equation is our main result.

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\Delta\phi$$

Notice that since $\cos(-\theta) = \cos(\theta)$ that it doesn't matter if you define $\Delta\phi = \phi_2 - \phi_1$, or $\Delta\phi = \phi_1 - \phi_2$.